THE

MATHEMATICAL GAZETTE

EDITED BY

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THE WORK OF A JUNIOR MATHEMATICAL ASSOCIATION.*

By G. L. Parsons.

The Junior Mathematical Association of London, of which I expect very few of you have even heard, is an association in a sense of the word rather different from that in which it is used in the title of the Association I am now addressing. It is the name adopted by some eight large schools (of the Public School type) in and around London for a joint Mathematical Society which has been running amongst the Sixth Form mathematical specialists in those schools for nearly five years. It is essentially an association of institutions rather than of persons, and this type of organisation will at once be seen to be necessary by any one who considers the effect of the changing personnel of Sixth Forms. It possesses therefore no strict membership roll, no subscription, and, incidentally, no harassed treasurer. All the same, it is alive, as I hope to be able to show you by the end of this short paper. It might also be recorded at this point that a somewhat similar association exists amongst some of the Classical Sixths.

It would perhaps be best to relate first some of the events which led to the formation of this society. In February, 1930, Mr. A. C. Heath inveigled Professor Hardy to come and give a short lecture to his mathematical specialists at St. Paul's. The actual promise was exacted by some curious manoeuvre I have never fully understood, but at any rate we duly appreciated its efficacy when Mr. Heath kindly invited me to attend and to bring along the Sixth Formers from Merchant Taylors'. After this lecture I suggested to Mr. Heath that we should consider getting together several schools in this way and he gladly concurred. Accordingly, invitations were sent out to eight schools to attend a meeting at M.T.S. on 18th March,

* A paper and discussion at the Annual Meeting of the Mathematical Association, 8th January, 1935.

1930. At this meeting a very interesting paper was given by Mr. E. H. Smart of Birkbeck College on " Rolling Curves ". After this meeting the question of further meetings of the same kind was discussed and a committee, consisting of a master and a boy from each school, was convened for the next week. This committee duly met, and the lines laid down at this meeting determined the early progress of the idea.

The schools which sent representatives to the first meeting were City of London (Mr. C. G. Nobbs), Dulwich College (Mr. Boon was unable to come himself and sent Mr. Dockeray), Highgate (Mr. J. L. Thomas), Merchant Taylors', St. Paul's, and University College School (Mr. F. A. Hills). The total number at the meeting was just under fifty. Westminster were invited and expressed willingness, but pleaded inability, being unable to fit in the times, partly owing to the Tuesday-Thursday-Saturday system of half-days. King's College School, Wimbledon, inadvertently omitted from the list of schools invited at first, joined in at about the third meeting. More recently we have had the pleasure of welcoming representatives from Harrow, and of holding there the first lantern-lecture the association has ever had, on the interesting subject of " Babylonian Mathematics ", by Dr. Sidney Smith, from the British Museum. I should here say that these meetings have been held in all cases with the consent of the headmasters of the schools, and it is only fitting that a tribute should be paid to their interest and cooperation. This has enabled the meetings to be held in school-time—usually at 3 p.m.—and has also made possible a very pleasant mode of terminating the proceedings, with which the members of this Association are not unfamiliar—viz. the consumption of tea.

The objects of the Association, as discussed at the first committee meeting, were threefold :

(i) To increase interest in Mathematics by lectures on matters not necessarily included in the normal syllabus, and in general to stress the unity and variety of the subject.

(ii) To promote and stimulate interchange of ideas between the schools, both among masters and boys.

(iii) To endeavour to promote the research spirit, *e.g.* by entrusting some of the papers to be read by competent boys.

The experience gained in the five years of the life of the Association tends to show that objects (i) and (iii) are being well achieved. The second object is more intangible, and success is not so easily observed. As far as masters are concerned it is certainly successful : I should like to place on record here my personal appreciation of friendships formed and cemented at these meetings. But as to boys, any one here who has any experience of inter-school activities will realise the difficulty of getting any sort of free intercourse between boys of different schools (and I suppose girls are about the same !). We tried the system of wearing labels once, but that made everybody more self-conscious still—so we dropped it. All the same, I have heard old pupils now at Cambridge refer to meeting again, as under-

graduates, boys they had *seen* at J.M.A. meetings—and I suppose we must be content with that as a beginning.

It was decided to hold five meetings every year (two in each of the Winter Terms and one in the Summer). The question of a Summer outing has also been discussed at times but has never materialised. Meetings are held at the various schools in rotation. The officials of the Association were elected, consisting of a chairman and a boy secretary, who holds office for a year. His duties consist of the keeping of full minutes and the sending of notices. Replies to the notices are sent direct to the school which is holding the meeting. I am pleased to say that we have had a succession of very efficient secretaries, the first being K. H. Beales of Dulwich College, in proof of which statement the minute book is here and open to your inspection after the meeting. I am sure you will be glad to observe that the minutes of the last meeting (unsigned as yet) are already written up.

One of the suggestions made at the original committee meeting was that some distinguished mathematician who had received the earlier part of his education at one of these schools should be invited to be President. Sir J. H. Jeans, an old Merchant Taylor, very kindly consented to act in this capacity and has always taken a very keen interest in the proceedings, of which he receives full reports. He delivered a Presidential address in September, 1932, which might well serve as a model on account of its suitability for the audience to which it was addressed.

Meetings in general have been about equally divided between outside and boy lecturers. At first we thought that two boys would be necessary in order to fill the time—about $1\frac{1}{4}$-$1\frac{1}{2}$ hours, but this proved to be an illusion, and since then several papers covering the full time have been read by boys. The subjects of these papers can be found reported in the minute book—I quote a few: Gyroscopic Motion, Gamma Functions, Divergent Series, etc. We have also had the pleasure of hearing several distinguished outside speakers in addition to those whose names I have mentioned. I may say here that the Association is much indebted to them for their assistance. It would manifestly be impossible to run an organisation of this sort on lectures given by boys alone. The alternating of outside and boy lecturers has so far been quite successful. So far, little work in the way of papers has been done by the masters concerned, though I will plead guilty to having read in June, 1934, a paper, the subject matter of which was " cribbed " shamelessly from one given to the London Branch by Mr. W. C. Fletcher.

In October, 1930, the suggestion was made, I think by Mr. C. G. Nobbs, that an Essay Competition should be held annually. At first the idea was that a small prize should be awarded by the masters, but when the suggestion was reported, together with the other proceedings, to the President, he very kindly offered a handsome prize for five years. The President's Essay Prize has been a very successful feature of the Association's activities. It is con-

ducted as follows. Two subjects are chosen in November by the masters and submitted to the President for sanction. They are then announced to the schools, together with a list of books suggested for study. The essays are sent in in the following May and are adjudicated, usually by a kind outside helper, another group of friends of the Association whose services must not go unmentioned. Numerically the entries for this prize are not impressive—we rarely have more than eight—but to judge from the examiners' reports this is more than compensated by the quality offered. The prize has usually been divided. To give an idea of the scope of the essays, I quote the subjects for 1932. They were: Non-Euclidean Geometry, and The Motion of the Planets. As I am sure that many of my hearers will think that these subjects, especially the first, are too difficult for any good work to be produced by boys, I should like to quote a few words from the examiner's report for that year. The examiner was Professor G. B. Jeffery, F.R.S. He says: "... I have found it very difficult to decide as to which of these essays most merits the prize. Every one of them has good points, and four or five are really excellent. Most of the essays give a clear and sufficient exposition of the mathematical theory. I particularly commend, however, those parts of the essays which attempt to trace the growth of ideas behind the mathematical development. There are a number of passages of this sort which impress me as giving evidence of maturity and real discernment ...." It is, of course, superfluous to add that no assistance is given to boys in preparing their essays, beyond the provision of books. Where these are not available in school libraries, efforts have been made to get them added, and failing this, private copies have been lent, but except for this the work has been done independently. The essays are not seen, as far as I know, by anybody until they are handed in and sealed up to be sent to the chairman.

As a variation from set lectures, successful Problem Drives have been held, the first being on 28th January, 1932. The details of these were first worked out by Mr. Heath, who brought to the task a long experience as an organiser of Scout activities. Small prizes have usually been given by the masters for the best (and worst) performances. At least when I say "worst" I believe we usually give a prize for the lowest prime number of marks—just to mark our mathematical status! These afternoons involve a good deal of work on the part of the organisers, especially in the setting of suitable problems. I hope to have an opportunity of saying more about this at the London Branch on 23rd March. Unfortunately I have not a set of past problems by me here, but, in conjunction with Mr. Styler, I am responsible for producing some by 31st January.

As far as it is legitimate to draw any general conclusions from so short a trial as five years, we may safely say that a good measure of success has attended the venture up to the present time and that there is good ground for hoping that similar progress will be made in the future. I think it is everybody's experience that Mathe-

matical Sixth Forms have been a little below par recently. This is no doubt due to economic factors rather than to inherent deterioration in the available material; but the point here is that if an association of this kind can keep going in the leaner years there is some chance that it will continue to flourish in the future. All who have been present have been interested in the meetings, though some have been a little too difficult. I well remember an orgy of model-making which followed Mr. W. C. Fletcher's talk on Solid figures and also some heated discussions on the subject of the angles in the Great Stellated Dodecahedron. Perhaps our subjects have not always strayed so far from the regular curriculum as we should have liked. This is partly due to the lack of easily-assimilated books on the more obscure parts of mathematics (where, for example, can we find a simple and readable account of the hypotheses connected with the resistance of the air or of the mathematical principles of aviation ?) and partly to the difficulty of finding qualified lecturers. It occurs to me at this point that here is a real chance for the Mathematical Association. Many members of the Association no doubt have special knowledge of some one particular branch of work which is not familiar to schoolmasters or schoolboys. No very deep knowledge is really needed; in fact we need more those who realise the difficulties of the initial stages rather than those with a profound grasp of the most advanced parts. Subjects which occur to my mind are Aerodynamics, Ballistics, Map Projections, but there must be many others. It would be of the very greatest assistance to those who have to run this and similar educational ventures if they had the assurance that qualified and willing expert advice and help could be obtained—and I am afraid I must add—gratis! I do not wish to say that we have not had in the past the greatest encouragement from some senior members of this Association. Mr. Boon, for example, has always been behind us with advice and help whenever called upon, but perhaps the time has come to make a wider appeal.

It may perhaps be felt in some quarters that these meetings and other activities take up valuable time which could be more usefully employed on those parts of mathematical learning which are included in the syllabuses of the regular examining bodies. This is part of the utilitarian heresy with which all schoolmasters are familiar, and into which they themselves occasionally fall. I would say at once that, as far as the meetings of the Association are concerned, we repudiate that heresy with all the conviction we can command. So far from noticing any adverse effect upon Higher Certificate and Scholarship results during the time that this experiment has been in progress I think we can claim to have sent to the Universities more and better scholars.

In conclusion, let me say that we all hope that other schools or groups of schools will try similar experiments for themselves. Naturally, local conditions vary a great deal, and it is not possible to lay down lines for schools in London which are likely to be the successful method of tackling a problem for, say, schools in the West Riding of

Yorkshire. That is manifestly impossible. Nevertheless, I hope that others will attempt the experiment to see what they can make of it; and I am sure I am speaking for all my colleagues at the other schools concerned in the experiment I have tried to describe to you this morning when I say that the results of our experiences are freely at your disposal if you care to call upon them. We feel ourselves that they have been worth while, and we are quite convinced that the same experience will be the lot of any others who try the same experiment.

DISCUSSION.

Mr. F. C. Boon (Dulwich College) thought it would be a pity if those hearing for the first time of the Junior Mathematical Association of London went away with the idea that the problems were of the nature of scholarship papers. He would therefore like Mr. Parsons to say what was the nature of the problems set, the number set, and also the time allowed for solution.

Mr. Parsons thought he could not do better than quote from the minutes of the first meeting of the Junior Mathematical Association at which a Problem Drive was held. It took place at University College School on Thursday, 28th January, 1932, and, the minutes of the previous meeting having been confirmed, "The Association then proceeded to hold a new method of mathematical entertainment for the afternoon: a Problem Drive arranged by Messrs. A. C. Heath, C. G. Nobbs and N. R. C. Dockeray. Seven tables were held, six sitting at each table; at each table problems were set, the time allowed to do them being six minutes. After each period of six minutes, three from each table moved up to the next table and three moved down. Many interesting problems were set and a very entertaining and enjoyable afternoon was spent. After everyone had been to the seven tables in turn, two prizes were given to those with the highest scores. The first prize was won by N. Davis (City of London School) who beat A. H. Gould (University College School) by half a point. A prize was also given to the person with the lowest prime number of points, the prize appropriately being a book entitled *Can you Solve it?*" The problems had necessarily to be some short snippets, and were arranged on little pieces of paper at the tables. For example, one paper consisted of finding a number of summations of the well-known Σ kind graduated in difficulty, running up to something fairly hard. The problems were all very short and could be got out, once the method was realised, in two or three minutes.

The President referred those who wanted ideas as to lectures to the reports of the American Undergraduate Mathematical Clubs which had appeared in the *American Mathematical Monthly*. Two or three years ago Professor Archibald of Brown University, Rhode Island, ran a series throughout a year, and there were heavily documented notes on all sorts of familiar problems for clubs of that nature. He did not think members could do better than look up

that series of papers if they needed suggestions for subjects and references to literature. Professor Archibald had sent offprints of his reports to the library, but on looking through the set some months ago the speaker had discovered gaps, which Professor Archibald had since been kind enough to fill from his own stock. Thus there was now available for members a complete set of offprints for that year bound into one volume, as well as the papers available in the volume in which they had appeared.

Mr. W. F. Bushell (Birkenhead) paid a tribute to the admirable and interesting paper read by Mr. Parsons and expressed appreciation of the experiment being carried on. He ventured to ask one or two questions. Did boys who were going to enter for Science Scholarships attend these meetings, as well as boys who were candidates for Mathematical Scholarships at the Universities ? Further, Mr. Parsons had spoken of their essay competitions, and named a pair of subjects, one pure and one applied, in a particular year. Were the boys given a wider choice, say half a dozen subjects, or not, to choose from ?

Mr. Parsons replied that those who attended the meetings varied, but, in general, they were mathematical specialists, and that meant, in London schools at any rate, not only those who were going in for Mathematical Scholarships at the older Universities but a certain proportion who might be going up to London and also a goodly number of intending actuaries. There were many occasions on which science specialists had attended the meetings. In some schools there was not any rigid line of demarcation between mathematicians and science specialists, so intending scientists also attended the meetings. Finally, the subjects for the Essay Competition were bound by the constitution of the Association to be two in number and usually they were chosen so that one was pure and the other applied mathematics, but the boys were free to choose which they would take.

Mr. A. C. Heath (St. Paul's) added that on one occasion at least the Essay Prize was won by a Science specialist, the subject being connected with the theory of internal combustion engines. In regard to the Problem Drives, one set of questions was entitled " Cautionary Tails "—" To what set of theorems do the following restrictions apply ? " A Problem Drive should not be too serious. There would be a possible gap of two years between boys who had just won a Scholarship and those who had just joined a Sixth Form ; so the questions should not be too difficult or too deep, or the younger boys would be discouraged by zero scores at many of the tables.

Asked by **Mr. Hope-Jones** for a specimen of a Cautionary Tail, **Mr. Heath** instanced such questions as : " To what theorems do the following restrictions apply ?—x must be less than 1, provided no three of the points are in a straight line, etc." There would be several answers to each question ; the more knowledge one had the more theorems one could fit to the Cautionary Tail. Mr. Nobbs originated this idea.

The President asked, was the Junior Mathematical Association of London now a closed association, so to speak, or was it hoped to attract other London schools into it, to which **Mr. Parsons** replied that there must be a natural bound to the size of such an association. For instance, not all schools possessed large lecture rooms, and from that point of view the meetings had at times rather strained the bounds. Again, there was the question of expense, and the work in connection with the meetings. It was felt, and he thought Mr. Heath would agree, that about fifty to sixty was the ideal size for the meetings of such an association; otherwise, they became straggly and the boys had not an opportunity of talking to each other.

Mr. Heath said there had been held another type of meeting which had proved successful: instead of a boy giving a lecture lasting forty-five minutes, which involved much research, there had been quarter-hour talks on some fairly simple subject. For instance, the evaluation of π, magic squares, lightning calculations; such talks gave more boys a chance of taking part, and they afterwards resulted in a good deal of argument over the tea-table; and that was very good indeed.

Mr. Parsons regretted that he had omitted mention of the meeting to which Mr. Heath had referred which was held at Dulwich College on 27th November, 1933, when the speakers were R. H. Wharton (Dulwich College) on the "Evaluation of π", F. B. Atkinson (St. Paul's) on "Magic Squares", K. Sisson (Harrow School) on "The Calculus of Operators", and K. J. L. Jamieson (Highgate School) on "Lightning Calculations".

Mr. M. P. Meshenberg (Tiffin's School) said the mention of the attendance of intending actuaries at the meetings raised in his mind the question as to how many other schools had those intending actuaries and what special provisions were being made for them in the schools.

The President felt, if there was anything else to be said upon the subject of the paper, it would be better to postpone that interesting question, whereupon **Mr. Parsons** thought—as half answer to the question—he might say that the Junior Mathematical Association had at least once prevailed upon someone to do something for actuaries. Mr. H. Freeman had come from the Institute of Actuaries to lecture to the Association on 8th November, 1930.

Mr. Heath added that another such lecture was that by Dr. J. Henderson on "Mathematical Statistics", which lecture had even interested the biologists and historians in the school.

There being no further questions, the meeting, on the proposition of **The President**, accorded a hearty vote of thanks to Mr. Parsons for his interesting paper, the President remarking that it was very seldom one could say that a paper contained nothing that the veterans had heard twenty or more years ago; Mr. Parsons had submitted something quite new.

THE BEARING OF STATISTICAL AND QUANTUM MECHANICS ON SCHOOL WORK.*

By D. R. Hartree.

As the President has told you, the subject of this lecture was not my own choice. When first asked to speak on it my impression was that, as the phrase goes, the answer was a lemon ; but, on thinking it over further, it seemed that quantum mechanics might have some bearing on school work, although that influence is in a sense only accidental, as I will explain. As far as I can see, however, statistical mechanics, the other half of the subject of the paper, has no bearing on school work, and it only occurs in the title because that title was given me !

I do not think that the inclusion of quantum mechanics as such in a school mathematics course is either practicable or desirable. It is not practicable because it is impossible to get very far in quantum mechanics without a considerable knowledge of differential equations—even the motion of a particle under a constant force, if treated exactly, involves Bessel functions of order $\frac{1}{3}$; or if you do not want to have differential equations, you must be prepared to have algebra with non-commutative multiplication, which is also rather severe on the school pupil. Moreover, it is not desirable, because by the end of the school course the pupils hardly have an adequate physical background to start the mathematical development of the subject ; one should not talk of the diffraction and interference phenomena of electron beams before the pupils have come across diffraction and interference phenomena in simpler cases of wave-motion, such as water or sound waves or in optics ; and that perhaps only comes towards the scholarship part of a science course in the school. And I think it most undesirable to try to develop the mathematical side of any branch of applied mathematics before the physical foundations are fully laid, because this tends to give quite a wrong balance to the subject. Therefore, the direct bearing of quantum mechanics on school mathematics work is, in my opinion, nil. The possibility of including an introduction to the physical ideas in a physics course is another matter ; but I take it that this meeting is concerned with the subject as it affects a mathematics course.

It has, however, an indirect bearing, in that the development of quantum mechanics, like that of the theory of relativity, has pointed out certain assumptions that lie at the basis of classical dynamics, and which most of us probably never recognised before quantum mechanics pointed them out. It is just a historical accident that those assumptions have been revealed by quantum mechanics, and it is because it has revealed them, and only for this reason, that it seems to me to have a possible bearing on school mathematical

* A paper at the Annual Meeting of the Mathematical Association, 8th January, 1935.

work, and all I can do without departing from the subject on which I was asked to speak is to talk, not about quantum mechanics, but about these assumptions in classical dynamics on which the quantum theory has thrown a light.

Whether one should bring into school work even the points I am going to discuss I leave others more experienced than myself in school teaching to determine; personally, I rather doubt it. The points which arise depend on discussion of the concept of velocity at an instant, and one must first establish the concepts of classical dynamics, and get them thoroughly into the minds of pupils, before they can appreciate where the crucial assumptions in those concepts lie. And after all, classical dynamics is valid for all practical purposes for almost any phenomenon on the macroscopic scale, that is, on the scale of matter as one handles it in everyday life; it is only on the atomic scale that the effects of the underlying assumptions become serious, and it is only on the macroscopic scale that students usually want to make applications of dynamics in their ordinary course; only a few go on to need a knowledge of dynamics on the atomic scale. A discussion of the assumptions underlying the concept of velocity and so on, if it came into the school course at all, would come somewhere near the end and would have to be something of a digression, but I think it probably better for it to be in the beginnings of the University course, where one should start from the beginning again and discuss the concepts of classical dynamics from scratch, rather than in the school course.

When it was realised that Newtonian dynamics failed to account for a certain range of phenomena on the small scale, the natural tendency was to seek the solution of the difficulty in the laws of motion or in the ideas of what forces there were acting on the bodies under discussion. It was some time before it was realised definitely that there was no salvation that way, and some time still after that before it was realised where the difficulty really lay. It did not lie in the laws of motion or in the forces we think of as acting on bodies, but was really more fundamental still in the idea of velocity and the possibility of its measurement.

The idea of velocity at an instant is usually introduced through the idea of average velocity over an interval. If δx is the distance described in a time-interval δt, the average velocity in that interval is the ratio $\delta x/\delta t$, and we define the velocity v at an instant as the limit of that ratio as $\delta t \to 0$, usually without giving any serious thought to the question whether that limit in fact exists or not.

The first thing I want to do is to make you a little sceptical whether that limit does exist at all and therefore whether we have any justification for speaking of velocity at an instant. Those quantities δx, δt are measures of space- and time-intervals, and if that limit is to mean anything we must imagine that, at least ideally, we could carry out measurements of indefinitely small space- and time-intervals and verify that the value of $\delta x/\delta t$ differed from the value v by as little as we chose, for all δt less than a certain value; and

clearly that involves the question of whether we can, even ideally, measure indefinitely small space- and time-intervals to an indefinitely high accuracy.

I think that putting the matter in this way is enough to show you that at least it is not obvious that we can establish that the limit exists at all. I do not say definitely that we cannot; all I want to say is that it is not at all obvious that we can. After all, any material body whose motion we study, and the scale by which we measure its displacement x, are made up of atoms; for measurement of very small intervals we want the finest markings we can get, and the smallest mark we can put on the moving body as a mark of reference is a single atom, and the smallest graduation we can put on the scale is a single atom. Now atoms of a solid are not at rest, even at absolute zero of temperature; they are fluctuating about mean positions with a displacement of the order of 10^{-9} cm. Therefore if we try to approach the limit $\delta x/\delta t$ as δt tends to zero, then as soon as δx becomes of the order of 10^{-9} cm. the fluctuations in the positions of the graduations of the scale and the mark of reference we put on the body are of the same order as δx. Therefore when we use successively smaller and smaller time-intervals what we can expect to get is something like Fig. 1.

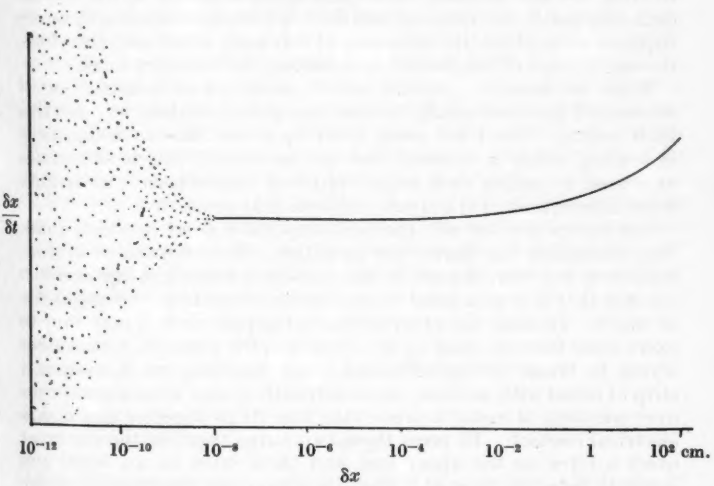


FIG. 1.

Suppose we take different time-intervals δt in the neighbourhood of a definite instant in the course of the motion of the body we are considering, and plot $\delta x/\delta t$ against δx. Then $\delta x/\delta t$ will possibly have a variation with the space-interval δx when that is very large

because of the large-scale variation of the velocity of the particle. For smaller δt, the values of $\delta x/\delta t$ will steady down and appear to be tending to a limit. But when δt is so small that δx is of the order of 10^{-8} cm., the values of $\delta x/\delta t$ will begin to vary again, and in an irregular way, because the measured δx will depend upon the displacements of the particles which we take as reference marks on the scale and on the body, at the instants of observation, from their mean positions. Therefore we would get not a smooth curve going on, but the various observations would give a scattered distribution of the values of $\delta x/\delta t$, which become rapidly more scattered as the time- and space-intervals become still smaller.

These scattered points represent possible results of observations we should make on the body, using our scale graduations, and a mark of reference on the body, which are of atomic size and partake of the motion of the atoms of the body. It is not possible to get away from this difficulty by saying: "Take the average over a certain time-interval of the positions of the marks on the body and the marks on the scale", because the whole point of the definition of velocity at an instant which we are trying to establish is that x and $x+\delta x$ are the measures of the position at definite instants t and $t+\delta t$. The question, quite apart from quantum mechanics, is whether we can measure time- and space-intervals which are indefinitely small, yet with the indefinitely high accuracies such as are required to establish the existence of this limit at all and, therefore, the significance of the definition of velocity in the strict sense.

What we mean in practice by "velocity at an instant" is, of course, not this limit at all, because it is quite uncertain whether this limit exists. What we mean is the *apparent* limit to which $\delta x/\delta t$ is tending when δt is small but *not too small*; this is sometimes expressed by calling such δx, δt "physical differentials", as distinct from differentials in the strict mathematical sense.

But that is not actually the most important point to which quantum mechanics has drawn our attention. More important is this: supposing you have a particle moving along a straight line and you observe that it is at a point whose displacement from the origin is x at time t. To make the observation on the particle it is necessary to exert some force or other on it. Consider, for example, a motor-car trying to break the speed record; the recording mechanism is a strip of metal with another one underneath it, and when the car runs over one strip of metal it presses the two strips together and makes electrical contact. To press these two strips together the car must exert a force on the upper one, and there must be an equal and opposite force on the car. That is to say, the observation of the position of the car involves a reaction on the car itself. Clearly, on any macroscopic scale that reaction can be made so small as to be negligible in practice, but the question arises: can we ever make it zero? Because when we want to measure very small time- and space-intervals, as we must if we want to try and establish the existence of a limit of $\delta x/\delta t$ in the strict sense, this reaction may become

serious, and this is where quantum mechanics throws a definite new light on the subject. It tells us quite definitely that we cannot make that reaction zero; not only that, it also tells us that the more accurately we try to measure the position of a particle, the bigger is the reaction of the observation on the particle.

To establish that the limit exists, we want, as I have said, to be able to measure δx and δt, however small they are. But that is just what quantum mechanics says we cannot do, because if we want δx very accurately we must have x at time t with a similar accuracy, and the more accurately we try to measure x at time t, the more we disturb the subsequent motion of the particle, so that though we might try to approach the limit and make δx and δt very small, the subsequent motion of the particle would be completely different from what it would be if we did not observe it.

That is a rough statement of the Uncertainty Principle of Heisenberg, and that, again, raises doubts whether the velocity, in the strict sense of the limit of $\delta x/\delta t$, exists at all as a physically determinable quantity.

There is a third way in which quantum mechanics may affect the idea of velocity; that is, in the emphasis which it puts on the question of the identification of the moving particle. When an astronomer is observing the position of the moon in the sky, he observes it one night, then the next, and has no doubt that it is the same moon in both cases. Why is he so sure that it is the same? Roughly speaking, he knows that, ideally, he could keep his eye on it all the time and see that it was the same; he could follow it as the earth went round and see it was the same moon he was observing all the time. Also, as far as his evidence goes, there is only one moon; there are not several indistinguishable moons, and therefore, if he sees a moon, he knows it is the same one each time.

But now suppose we try to study the motions of the electrons in an atom containing several electrons. Two electrons are not only as like as two peas; they are very much more like, in fact they are completely indistinguishable. Therefore one gets this sort of question arising (Fig. 2):

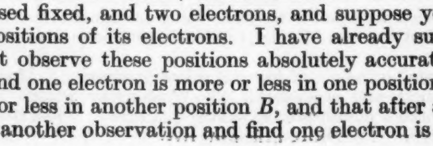


FIG. 2.

Consider a helium atom, which consists of a nucleus which can be supposed fixed, and two electrons, and suppose you try to observe the positions of its electrons. I have already suggested that you cannot observe these positions absolutely accurately; but suppose you find one electron is more or less in one position A and the other more or less in another position B, and that after an interval δt you make another observation and find one electron is somewhere about

C and the other somewhere about D, how do you know whether the one at C in the second observation is the one at A or at B in the first? You must not imagine that you could keep your eye on the electrons in the intermediate stages of the motion, in the way that the astronomer could, ideally, keep his eye on the moon all the time; if you tried to do this with the electrons, such an observation would blow them right out of the atom: and the answer is that you do not know. And it is clear that the velocities by which you describe the motion of these two electrons will be completely different according as you think that the one at C at the second observation is the same as the one at A or at B in the first.

Quantum mechanics lays considerable stress on the question of impossibility of identifying the individual particles in a system consisting of a number of indistinguishable particles. It is one of the most striking successes of quantum mechanics that it has shown that the forces binding the atoms in a chemical molecule are exactly of the same type as the electrical forces which are involved in other physical phenomena: that it is not necessary to introduce any new kinds of " chemical forces " in order to give an account of chemical combination. A large part of the quantum theory of chemical combination, and also the quantum theory of the magnetic properties of iron and similar substances, is based on the mathematical formulation of this phenomenon of the indistinguishability of two electrons. That is the third point in regard to which quantum mechanics tells us we have to be careful in the ideas which really lie at the base of kinematics, the geometrical study of the motions of particles, before we go on to talk of dynamics at all.

To sum up, there are three assumptions at the basis of dynamics which were probably only tacitly recognised, if recognised at all, before. Firstly, there is the assumption of the possibility of measuring space- and time-intervals to an indefinite accuracy. Secondly, there is the assumption that the observation of a moving body does not affect its subsequent motion. Thirdly, there is the assumption of the possibility of identifying the moving body whose motion we are studying and being sure that if we observe it at two different times it is really the same body that we are observing. And it is in pointing out and underlining those assumptions that the bearing of quantum mechanics on school work, if any, depends. D. R. H.

GLEANINGS FAR AND NEAR.

1003. "Things which are equal to the same thing are equal to each other." This was one of the first axioms of Euclid, but Euclid was certainly not a cereal chemist.

... the fundamental principle of integral calculus, namely, " If you take enough nothings, their sum is something."—Washington Platt, " Higher versus lower mathematics in interpreting baking quantity ", *Cereal Chemistry*, 10 (1933), pp. 215-216. [Per Dr. J. Wishart.]

ON DIFFERENTIALS.

By D. K. Picken.

1. The article on " Differentials ", in the May *Gazette*—based on a discussion at the last Annual Meeting of the Association, and referring back to articles of July 1931 and February 1932 *—has so clarified certain issues for the writer as to confirm a long-standing conviction that the standard definition of " differentials " is quite unsound, and that the view set out below (which he had developed in his own teaching †) is much truer to all the facts of the situation.

The crux of the position is, of course, the definition of dx to mean Δx—and of dy, therefrom, so as *not* to mean Δy (except when y is constant or equal to x). This, among other things, involves an arbitrary distinction between *argument* (or, *independent* variable) and *function* (or, *dependent* variable), which cannot properly be imposed upon functional relations—and cannot, in fact, be maintained.‡ The arbitrariness of the definitions is made explicit by the explanation that " differentials " mean " simply two quantities whose ratio (is) equal to the derivative of $f(x)$, etc." § Actually, in matters of such fundamental importance as this, it is almost inconceivable that any such arbitrariness should be admissible.

Professor Temple's contribution to the discussion || is of special interest, as making clear the sound mathematical principles by which any *quantitative* conception of " differentials " would have to be determined. But a " quantitative " conception is, in fact, unnecessary; and the burden of this article is a simple, commonsense approach to the subject—which is more or less equivalent to that by way of the Theory of Nul Sequences.

2. The origin of the discussion was introduced by reference to " the notation of differentials . . . invented by Leibniz in 1675, at about the same time as Newton invented his ' fluxions ' ".¶ That is the point to which to go back. Each of the two great philosophers, who " discovered " the Infinitesimal Calculus, " invented " a notation which has stood the test of time and use; but that of Leibniz stands out as one of the greatest instances of intuition of genius in all the history of this science of ours, which lives and moves by good notations: ** an instance of a notation so perfectly adapted to its

* *Gazette*, No. 214, pp. 401-3; 217, pp. 5-10; 228, pp. 68-79.

† See Note 686 on " The Notation of the Calculus ", *Gazette*, No. 166, p. 387 (Oct. 1923)—where some of the points dealt with below are summarized.

‡ See, *e.g.* Professor Wilton's point, at the foot of p. 5, No. 217.

§ No. 228, p. 77 (J. T. Combridge); also, p. 78.

|| No. 228, pp. 68-70.

¶ No. 214, p. 401.

** Compare, the standard notation for the Natural Numbers—as another perfect notation, which, however, took centuries to evolve; and, as an example of the opposite kind, the very imperfect notation for " roots " and " logarithms " (and terminology of the mathematical number-system).

purpose that it has proved exactly right for needs which could not possibly have been anticipated by its inventor—in a theory which developed and expanded so amazingly in the ensuing centuries.

The essence of this notation is that dy/dx is used for $\lim \Delta y/\Delta x$,* in which $\Delta x, \Delta y$ denote the corresponding infinitesimal † increment-variables determined from a given functional relation between x and y, so that

$$y = f(x) \quad \text{and} \quad y + \Delta y = f(x + \Delta x),$$

or, again—using the more general functional form—

$$f(x, y) = 0 \quad \text{and} \quad f(x + \Delta x, y + \Delta y) = 0.$$

The importance of this " differential " notation, for the limit in question, turns on the importance of the " difference " notation (" Δ ") upon which it is based; and it seems necessary to digress upon this, in passing, as the discussion casts some doubt upon the importance of the Δ-notation. The problem of the Differential Calculus is that of the quotient of corresponding differences $(y_2 - y_1)/(x_2 - x_1)$—for functionally related variables x, y—in the " limiting " case of " coincidence " of x_2, y_2 with x_1, y_1; *i.e.* it is the problem of the limiting value of the quotient $(y - y_1)/(x - x_1)$, in which x_1, y_1 are given (corresponding) values of x, y and $(x, y) \to (x_1, y_1)$. This limiting value depends upon x_1, and upon the functional relation between x and y. But the actual value, x_1, of x, in question, is not, in general, significant; the result obtained is " general ", for the range of possible values of x. Hence, it is better mathematics to replace (x_1, y_1) by (x, y)—provided, of course, we then replace (x, y), in the quotient, by something else that is appropriate; and for that purpose $(x + \Delta x, y + \Delta y)$ proves to be the perfect notation. It is a characteristic mathematical practice, of taking two steps at once (by a leap in thought)—where, actually, one of the steps is essentially subsequent to the other.‡

The quotient $\Delta y/\Delta x$ may then be regarded as a function of the two arguments x and Δx; and its limit, when $\Delta x \to 0$, is then the function of x which we call " the derived function "—properly denoted by the Newtonian symbol y'. Thus

$$y' = \lim (\Delta y/\Delta x) = dy/dx;$$

and if we have a given *expression*, $f(x)$, for the *function* y, we obtain the corresponding expression, $f'(x)$, for the derived function, from

$$f'(x) = \lim \{(f(x + \Delta x) - f(x))/\Delta x\};$$

* This is exactly the opposite view to that which is expressed at the top of p. 402 of No. 214: the difference being characteristic of the two different ways of approaching the subject of " differentials ". (See also Professor Wilton, No. 217, p. 5.)

† An " infinitesimal " is a *variable* used in a " limit " proposition, which is characterized by the fact that that variable $\to 0$, " in the limit ".

‡ It is instructive to draw three diagrams: (1) for two given points P_1, P_2; (2) for a given point, P_1, and the variable point, P, of the graph of y; (3) for two variable points P, Q of that graph.

and it is for this *expression*, $f'(x)$, that the term " derivative " may best be used.*

Thus, dy/dx is to be thought of as, ***not itself actually a quotient, but*** a function of x (or of y) which is ***the limit of a certain quotient***.† Corresponding to the notation, the term " differential quotient "—as used by Continental writers—is appropriate (meaning *limiting quotient of differences*).

And, correspondingly, the " differential " symbol " d ", in all its legitimate uses, may be interpreted to mean " $\lim \Delta$ ": as in the following discussion.

3. The justification for the Leibniz notation, d^2y/dx^2, etc., for " the higher derived functions ", is that these expressions actually mean $\lim \Delta^2 y/\Delta x^2$,‡ etc.—where

$$\Delta^2 y = \Delta(\Delta y) = f(x+2.\Delta x) - 2.f(x+\Delta x) + f(x), \text{ etc.}$$

4. The above definition of dy/dx implies that

$$\Delta y/\Delta x = y' + \epsilon = f'(x) + \epsilon,$$

where ϵ denotes a function of x and Δx, which $\to 0$ when $\Delta x \to 0$; hence that

$$\Delta y = y'.\Delta x + \omega = f'(x).\Delta x + \omega,$$

where ω is " an infinitesimal of higher order " than Δx (*i.e.* it is such that $\omega/\Delta x \to 0$ when $\Delta x \to 0$).

It is this *quantitative* relation (in " infinitesimal " variables) which may properly—and very usefully—be expressed in the " differential " form

$$dy = y'.dx = f'(x).dx$$

—which is to be interpreted as meaning that the infinitesimals Δy and $y'.\Delta x$ (to which it essentially refers back) are " ultimately equivalent ", in the true Newtonian sense (*i.e.* they are in " ultimate " *ratio* of equality).

As this is (by general consent) the critical point of the theory of Differentials, it is to be emphasized that, in the view here being presented, dx and dy are *not quantities*—and that dx definitely does not mean the same thing as Δx (any more than dy means the same thing as Δy). *These " differentials " are simply a part of the nomenclature of the Infinitesimal Calculus—always to be interpreted in terms of " limit " propositions, referring back to the infinitesimals* Δx, Δy.

* It will be noted wherein, on the one hand, I agree—and, on the other hand, I do not agree—with Professor Wilton (No. 217, p. 7) and others, on these points.

† The usage has a remote resemblance to that of such " names " as " cm/sec " for the " derived unit " of speed: that unit being not itself actually a quotient, but based on a relationship (of proportion) which is expressible by means of an actual quotient.

‡ Certain conditions (commonly satisfied) are necessary, in order that this limit-expression should give the second derived function, y''.

In particular, operational processes applied to them have such reference back to the parent quantities.*

5. The elementary relations of Differential Geometry yield at once to this approach. Thus

$$ds^2 = dx^2 + dy^2 = dr^2 + r^2 . d\theta^2$$

are consequences of the ultimate equivalence of arc-element and chord-element (measure γ), and the facts that

$$\gamma^2 = \Delta x^2 + \Delta y^2, \text{ and is ultimately equivalent to } \Delta r^2 + r^2 . \Delta\theta^2.$$

And, from the same infinitesimal right-angled triangles (using standard ψ and ϕ),

$$ds = dx/\cos\psi = dy/\sin\psi = dr/\cos\phi = r . d\theta/\sin\phi$$

—all with immediate " reference back " to the corresponding increment-infinitesimals (and to ultimate equivalence).

An interesting example of the method of differentials is that of the application of the " Polar " formulae, in well-known results obtainable from

$$p = r . \sin\phi \quad \text{and} \quad t = r . \cos\phi, \quad \text{using} \quad \theta + \phi = \psi ;$$

thus :

$$\begin{aligned} dp &= \sin\phi . dr + r . \cos\phi . d\phi \dagger \\ &= r . \cos\phi . d\theta + r . \cos\phi . d\phi \\ &= r . \cos\phi . d\psi = t . d\psi \\ &= r . (dr/ds) . d\psi = r . dr . d\psi/ds, \end{aligned}$$

whence

$$t = dp/d\psi \quad \text{and} \quad \rho = r . dr/dp ;$$

similarly,

$$\begin{aligned} dt &= \cos\phi . dr - r . \sin\phi . d\phi \\ &= \cos\phi . dr + r . \sin\phi . (d\theta - d\psi) \\ &= ds - p . d\psi, \end{aligned}$$

whence

$$\rho - p = dt/d\psi = d^2p/d\psi^2 ;$$

all the steps being justifiable by " reference back " to the increment-infinitesimals involved.

6. The use of the " differential " in the Integral Calculus can easily be brought into line.

Here the principle of a special " limit " notation is extended, in

* For example, dx^2 and d^2y in § 3—and all the detail of what follows. Compare, the operator " D "—which, though not itself a quantity, is subject to operational processes such as are primarily characteristic of quantities (and because of its relationship—of a different kind—to quantities). It must have been something like this idea of " differentials " that inspired Lewis Carroll's whimsical notion of the Cheshire Cat's disembodied smile !

† $y = u . v$ gives $\Delta y = u . \Delta v + v . \Delta u + \Delta u . \Delta v$, and, therefore, $dy = u . dv + v . du$; etc.

the use of "$\int$" for "lim Σ"—corresponding exactly to the use of "d" for "lim Δ":

$$\int_a^b f(x)\,.\,dx \quad \text{means} \quad \lim \Sigma f(x)\,.\,\Delta x,$$

in which the summation is specified in a standard way, the number (n) of terms $\to\infty$, and Δx is dependent on n in such a way that it $\to 0$ in each term.*

The substitution $x=\phi(u)$ † gives

$$\Delta x = \phi'(u)\,.\,\Delta u + \omega \quad \text{(as in § 4, above)};$$

and the term ω does not affect the *limit* of the summation. Hence

$$\lim \Sigma f(x)\,.\,\Delta x = \lim \Sigma F(u)\,.\,\phi'(u)\,.\,\Delta u,$$

i.e.

$$\int_a^b f(x)\,.\,dx = \int_a^\beta F(u)\,.\,\phi'(u)\,.\,du,$$

where

$$F(u)=f(x), \quad a=\phi(a), \quad b=\phi(\beta).†$$

The transformation of the definite integral is, therefore, properly specifiable by substituting

$$x=\phi(u), \quad dx=\phi'(u)\,.\,du.$$

For the Indefinite Integral: though an equivalent transformation can be deduced as "inverse" to the theorem on differentiation of "a function of a function", the process is much less direct and natural. Actually, one always uses the transformation in the above form—obtained directly from the Definite Integral; also in the surprisingly useful converse presentation

$$\int f\{\phi(x)\}\,.\,\phi'(x)\,.\,dx = \int f(u)\,.\,du, \quad \text{if} \quad u=\phi(x).$$

(The Definite Integral is, in fact, the true basis of the Integral Calculus; but that is a thesis which cannot be elaborated in this article.)

7. The method of Differentials is peculiarly applicable to functions of several variables ‡—as giving the only adequate way of stating the fundamental theorem on differentiation of such functions, viz. the theorem that

$$\text{if} \quad u=f(x, y, \ldots), \quad \text{then} \quad du=\Sigma f_x'(x, y, \ldots)\,.\,dx,$$

where $f_x'(x, y, \ldots)$, etc., are the several "partial" *derivatives* of the

* The writer gave a presentation of the detail here involved, in an article on "The Integral Calculus Theorem" in No. 61, pp. 5-7 (January 1907).

† For the immediate purpose of this discussion, there is no significant loss of generality in assuming that x varies monotonically from a to b as u varies monotonically from a to β: the general case being reducible to that case.

‡ The references to this application, in the *Gazette* articles, are rather slight, in view of its importance. In No. 214, p. 403, the approach is from the other point of view—to which reference has been made above in a footnote to § 2.

expression $f(x, y, \ldots)$ * : the proposition in differentials meaning that the increment-infinitesimal, Δu—determined by

$$u + \Delta u = f(x + \Delta x, y + \Delta y, \ldots)$$

—is " ultimately equivalent " (as defined in § 4, above) to the infinitesimal specified by $\Sigma f_x'(x, y, \ldots) . \Delta x$, in which Δx, $\Delta y, \ldots$ denote " arbitrary " increment-infinitesimals.

From this one theorem—so expressed—may readily be deduced all the great variety of results in this important field : *e.g.*

(1) If variables $x, y, \ldots$ are functionally related by

$$f(x, y, \ldots) = 0 \dagger,$$

then $$\Sigma f_x'(x, y, \ldots) . dx = 0 :$$

this meaning that the increment-infinitesimals $\Delta x, \Delta y, \ldots$—related by

$$f(x + \Delta x, y + \Delta y, \ldots) = 0$$

—are such that

$f_x'(x, y, \ldots) . \Delta x$ is " ultimately equivalent " to $-\Sigma f_y'(x, y, \ldots) . \Delta y$ (in which Δy, $\Delta z \ldots$ are arbitrary infinitesimals).

In particular, the general functional relation between *two* variables, viz. $f(x, y) = 0 \dagger$, gives

$$f_x'(x, y) . dx + f_y'(x, y) . dy = 0$$

—meaning that

$$f_x'(x, y) . \Delta x \quad \text{and} \quad -f_y'(x, y) . \Delta y$$

are " ultimately equivalent " infinitesimals ; whence

$$dy/dx = -f_x'(x, y)/f_y'(x, y).$$

(2) If u is a function of x, specified by an expression $f(x, y, z, \ldots)$, where $y, z, \ldots$ are functions of x, specified by $y = \phi(x)$, $z = \psi(x), \ldots$, then, using

$$du = f_x'(x, y, \ldots) . dx + \Sigma f_y'(x, y, \ldots) . dy, \quad dy = \phi'(x) . dx, \ldots$$

—as interpreted in this article—we get

$$du/dx = f_x'(x, y, \ldots) + \Sigma f_y'(x, y, \ldots) . \phi'(x)$$

—from which, again, we may proceed, in the same way, to derive expressions for d^2u/dx^2, $d^3u/dx^3, \ldots$.

8. The use of " differentials " in Differential Equations turns on these facts (of § 7).

The simplest, and most fundamental, type of differential equation (of first order and first degree) is expressible in the alternative forms

$$(1)\ dy/dx = f(x, y) \quad \text{and} \quad (2)\ P . dx + Q . dy = 0$$

* It is to be noted that in this part of the Infinitesimal Calculus the emphasis is on " expressions " rather than upon " functions ". Partial *derivative*, $f_x'(x, y, \ldots)$, has a perfectly definite meaning—in terms of a given *expression* $f(x, y, \ldots)$; partial *derived function*, $\partial u/\partial x$, may have many different meanings—according to the *expression* used for the function u. (See No. 166, p. 388.)

† It is important to note that $f(x, y, \ldots) = $ *constant* gives the same result. See § 8, below.

(where $P/Q = -f(x, y)$): the latter form being interpretable by the "ultimate equivalence" of $P.\Delta x$ and $-Q.\Delta y$. Its "complete primitive" is of the form

$$\psi(x, y) = C\ ;\ *$$

and the "differential" form, (2), of the differential equation is equivalent to

$$\psi_x'(x, y).dx + \psi_y'(x, y).dy = 0$$

—the principle of the "integrating factor" being thus the essential principle of the "integration".

The form

$$P.dx + Q.dy + R.dz + \ldots = 0,$$

in n variables, is "integrable" in so far as it is reducible to the above type, and is then expressible, by means of an integrating factor, in the form

$$\psi_x'(x, y, z, \ldots).dx + \psi_y'(x, y, z, \ldots).dy + \ldots = 0\ \dagger.$$

But the "differential" form for differential equations is the exception, rather than the rule: the form $f(x, y, y', \ldots) = 0$ being far more typical.

9. The power of this method of Differentials is remarkably exhibited by its use—as follows—in the envelope investigation.

Defining the envelope of the family of curves specified by

$$f(x, y, \lambda) = 0, \quad (\lambda \text{ the parameter}),$$

as a locus which touches, at each of its points, a member of the family, we have to investigate the locus—for variation of λ—of (x, y), such that the three *variables* x, y, λ satisfy

$$f(x, y, \lambda) \equiv 0$$

and so $\quad f_x'(x, y, \lambda).dx + f_y'(x, y, \lambda).dy + f_\lambda'(x, y, \lambda).d\lambda = 0$

or $\quad f_x'(x, y, \lambda).dx/d\lambda + f_y'(x, y, \lambda).dy/d\lambda + f_\lambda'(x, y, \lambda) = 0\ ;$

and (the condition of tangency)

$$dy/dx = -f_x'(x, y, \lambda)/f_y'(x, y, \lambda)$$

or $\quad f_x'(x, y, \lambda).dx + f_y'(x, y, \lambda).dy = 0$

or, again, $\quad f_x'(x, y, \lambda).dx/d\lambda + f_y'(x, y, \lambda).dy/d\lambda = 0\ ;$

such, therefore, that they satisfy

$$f(x, y, \lambda) = 0 \quad \text{and} \quad f_\lambda'(x, y, \lambda) = 0.$$

* An elementary argument, in graphical terms, for the "existence" of the primitive in this form is obtainable from $dy = f(x, y).dx$, by constructing a polygonal line from the approximate relation $\Delta y = f(x, y).\Delta x$—using an arbitrary initial point and an arbitrary Δx. The "limit" of this line, when $\Delta x \to 0$, is a curve which "satisfies" the differential equation; and there is *one* such curve through each point of the xy-plane.

† Such an article as that of No. 223, pp. 105-11 (Underwood), may be regarded as typical of the justifiable manipulation of the "differentials" in such cases (or, see any book on Differential Equations).

In seeking, conversely, to interpret (geometrically) this simple result, we use

$$f(x, y, \lambda) + f_\lambda'(x, y, \lambda) \,.\, \Delta\lambda = 0$$

—which, *in its limiting form*

$$f(x, y, \lambda) + f_\lambda'(x, y, \lambda) \,.\, d\lambda = 0,$$

may be written

$$f(x, y, \lambda + d\lambda) = 0\,^* ;$$

and we may thence state the two conditioning relations in the form

$$f(x, y, \lambda) = 0, \quad f(x, y, \lambda + d\lambda) = 0.$$

These determine a locus of (x, y), specifiable geometrically in either of the following ways :—

(1) locus of " *ultimate* intersection " of " consecutive " † members of the family ;

(2) locus of a point such that, for its values of x, y, the λ-equation $f(x, y, \lambda) = 0$ has equal roots—or, that two of the members of the family, on which the point lies, are " coincident ".

It is easy to see—from the specification (1)—that the envelope (if any) is included in this locus, but that it also includes any multiple-point locus of the family.

From the specification (2), it follows that the envelope is included also in the locus determined by

$$\phi(x, y, p) = 0, \quad \phi_p'(x, y, p) = 0, \quad (p = dy/dx)$$

—if $\phi(x, y, p) = 0$ be the differential equation of the family ; but *not* a multiple-point locus, except for multiple-points of the " cusp " type ; while, on the other hand, any " tac-locus " is covered by this " p-discriminant ", but not by the λ-discriminant.

10. This article does not deal directly with the question of what should be taught at the pre-university stage of mathematical teaching. It is concerned with the prior question of a sound approach to the particular subject under discussion. It has been sufficiently elaborated to demonstrate that a *non-quantitative* conception of Differentials—giving a useful (and even powerful) way of expressing limit results, from relations in terms of difference-*quantities*—is both adequate to all the facts and as simple as the essential subtlety of such ideas permits.

It ought, perhaps, to be specially noted that no theory—such as that of §§ 7, 8, 9 above—involving much use of partial differentiation, can be regarded as quite elementary.

Ormond College, University of Melbourne. D. K. Picken.

* Meaning that $f(x, y, \lambda + \Delta\lambda)$ gives, for such values of x, y, λ, an " infinitesimal of higher order " than $\Delta\lambda$.

† " Consecutive ", in terms of the λ-increment, $\Delta\lambda$—the tending to 0 of which gives the " ultimate intersection ".

DETERMINATION OF THE FOCUS AND DIRECTRIX OF A PARABOLA WHOSE EQUATION IS GIVEN WITH NUMERICAL COEFFICIENTS.

BY LAWRENCE CRAWFORD.

1. IN a note in the *Mathematical Gazette*, February 1934, pp. 43-46, I gave a method for determining the foci, etc., of a conic in general and showed that the method could be applied to the parabola. A direct solution for the parabola can, however, be put down more quickly.

2. If (α, β) is the focus and $lx+my+n=0$ the directrix, the equation of the parabola is

$$(x-\alpha)^2+(y-\beta)^2=\{(lx+my+n)/\surd(l^2+m^2)\}^2$$

or

$$(mx-ly)^2-2x\{(l^2+m^2)\alpha+ln\}$$
$$-2y\{(l^2+m^2)\beta+mn\}+(l^2+m^2)(\alpha^2+\beta^2)-n^2=0.$$

As the terms of the second degree in the equation of any parabola can be expressed as an exact square by multiplying by a constant, the equation of any parabola can be taken to be

$$(mx-ly)^2+2gx+2fy+c=0.$$

Thus l, m may be taken as known from the equation and the problem is to find n, α, β.

3. The equations for α, β, n are

$$(l^2+m^2)\alpha+ln=-g, \quad \text{..............................(i)}$$
$$(l^2+m^2)\beta+mn=-f, \quad \text{..........................(ii)}$$
$$(l^2+m^2)(\alpha^2+\beta^2)-n^2=c. \quad \text{.....................(iii)}$$

To solve for n, from (i) and (ii) we have

$$(l^2+m^2)^2(\alpha^2+\beta^2)=(ln+g)^2+(mn+f)^2,$$

and from (iii),

$$(l^2+m^2)^2(\alpha^2+\beta^2)=(l^2+m^2)(n^2+c);$$

hence

$$(ln+g)^2+(mn+f)^2=(l^2+m^2)(n^2+c),$$

and thus

$$2n(lg+mf)=(l^2+m^2)c-f^2-g^2,$$

so n is found.

Now

$$2(lg+mf)(l^2+m^2)\alpha=-l\{(l^2+m^2)c-f^2-g^2\}-2(lg+mf)g$$
$$=lf^2-lg^2-2mfg-l(l^2+m^2)c,$$

and

$$2(lg+mf)(l^2+m^2)\beta=-m\{(l^2+m^2)c-f^2-g^2\}-2(lg+mf)f$$
$$=mg^2-mf^2-2lfg-m(l^2+m^2)c.$$

Thus α, β are found and the focus is known.

The equation of the directrix is

$$2(lx+my)(lg+mf)+(l^2+m^2)c-f^2-g^2=0.$$

4. These results lead at once to the determination of other things of the parabola.

5. From equations (i) and (ii) in § 3,

$$(l^2+m^2)(m\alpha-l\beta)=lf-mg,$$

and so the focus lies on the line

$$(l^2+m^2)(mx-ly)=lf-mg.$$

But this line is perpendicular to the directrix, and, since it goes through the focus, this is the equation of the axis.

6. The latus rectum is twice the length of the perpendicular from the focus on the directrix and so is

$$2\{\pm(l\alpha+m\beta+n)/\surd(l^2+m^2)\}.$$

But multiplying equations (i) and (ii) in § 3 by l and m and adding, we get

$$(l^2+m^2)(l\alpha+m\beta+n)=-(lg+mf),$$

and hence the latus rectum is

$$\pm 2(lg+mf)/(l^2+m^2)^{\frac{3}{2}}.$$

7. The vertex is given by the equation of the axis,

$$(l^2+m^2)(mx-ly)=lf-mg,$$

and the equation of the parabola : it is therefore given by

$$mx-ly=(lf-mg)/(l^2+m^2)$$

and $$2gx+2fy+c=-(lf-mg)^2/(l^2+m^2)^2.$$

8. Take as a numerical example the parabola with equation

$$4x^2+12xy+9y^2+2x+2y+2=0.$$

Here $\quad m=2, \quad l=-3, \quad g=1, \quad f=1, \quad c=2.$

The equations for n, α, β are

$$13\alpha-3n=-1,$$
$$13\beta+2n=-1,$$
$$13(\alpha^2+\beta^2)-n^2=2.$$

Thus
$$\begin{aligned}169(\alpha^2+\beta^2)&=13n^2+26\\&=(3n-1)^2+(2n+1)^2\\&=13n^2-6n+4n+2,\end{aligned}$$

and $$n=-12.$$

The equation of the directrix is therefore

$$-3x+2y-12=0.$$

For the focus,

$$13\alpha = \quad 3n - 1 = -37\,;$$
$$13\beta = -2n - 1 = 23.$$

The focus is $\quad (-\tfrac{37}{13}, \tfrac{23}{13}).$

The axis is $\quad 13(2x+3y) = -3-2,$

or $\quad 26x + 39y + 5 = 0.$

The latus rectum is

$$\pm 2(-3+2)/13^{\frac{3}{2}}$$
$$= 2/13^{\frac{3}{2}}.$$

The vertex is given by

$$26x + 39y = -5,$$

and $\quad 2x + 2y + 2 = -(2x+3y)^2 = -\tfrac{25}{169},$

and thus the vertex is $\quad (-\tfrac{959}{338}, \tfrac{298}{169}).$

The tangent at the vertex is parallel to the directrix, and its equation is therefore

$$3(x + \tfrac{959}{338}) - 2(y - \tfrac{298}{169}) = 0,$$

or $\quad 3x - 2y + \tfrac{313}{26} = 0.$

The foot of the directrix is given by

$$3x - 2y = -12$$

and $\quad 2x + 3y = -\tfrac{5}{13},$

and so is the point $\quad (-\tfrac{478}{169}, \tfrac{297}{169}).$

That the vertex is the mid-point of the line joining this to the focus can immediately be verified. L. C.

1004. Besides, can you have forgotten that the gravest personages have given us light reading sometimes? (Foot-note: M. de Montucla, known for his admirable *History of Mathematics*, also compiled a *Dictionary of Alimentary Geography*; he showed me some fragments of it during my sojourn at Versailles.)—Brillat-Savarin, *Physiology of Taste* (English translation, 1925), p. 7.

This work appears to have remained a fragment; at any rate it is not in the British Museum catalogue. [Per Mr. F. P. White.]

1005. "And now this work." On a clean sheet of paper, Clovis-Abel wrote down William's first question. "Listen to me", he said; "forget the numbers and think only of the signs. You shall call plus 'my friend', and minus 'mine enemy'. And everything shall be clear."

And behold it was so.

For when Johnny came to multiply plus this by minus that, he translated under the Frog's guidance, saying: "The friend of my enemy is my enemy". Secondly, when William asked what $-15p \times -7q$ was equal to, they brushed aside these p's and q's, and chanted slowly together "The enemy of mine enemy is my friend". And then they worked out fifteen times seven. Johnny was entranced.—D. Wynne Wilson, *Early Closing*, ch. xix. p. 255. [Per Mr. I. FitzRoy Jones.]

TRIANGLE ET QUADRILATÈRE BORDÉS DE TRIANGLES ÉQUILATÉRAUX.

PAR V. THÉBAULT.

VOICI quelques compléments au remarquable article de M. N. M. Gibbins sur la figure constituée par un triangle bordé de triangles équilatéraux.*

1. Considérons un triangle BAC, et les triangles équilatéraux BCA_1, BC_1A, AB_1C et BA_3C, BAC_3, ACB_3 construits extérieurement puis intérieurement sur les côtés CB, BA, AC.

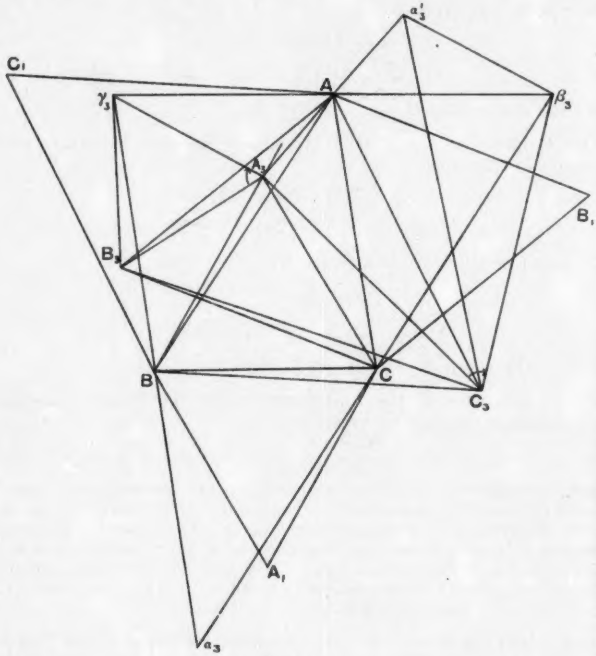


FIG. 1.

Théorème. Les sommets des triangles équilatéraux construits intérieurement sur les côtés du triangle $A_1B_1C_1$ et les sommets des triangles équilatéraux construits extérieurement sur les côtés du triangle $A_3B_3C_3$ coïncident avec les sommets du triangle anticomplémentaire du triangle ABC.

Soient α_1', β_1', γ_1' et α_3, β_3, γ_3 les sommets des triangles équi-

* *Gazette*, XVIII, May 1934, p. 95.

latéraux relatifs aux triangles $A_1B_1C_1$ et $A_3B_3C_3$, puis α_3', β_3', γ_3' les sommets des triangles équilatéraux construits intérieurement sur les côtés du triangle $A_3B_3C_3$.

Les triangles $B_3A_3C_3$ et $C_3\alpha_3'\beta_3$, par exemple, sont égaux, puisque

$$\begin{aligned} C_3\alpha_3' &= B_3C_3, \quad C_3\beta_3 = C_3A_3, \\ \angle\alpha_3'C_3\beta_3 &= \angle A_3C_3\beta_3 - \angle A_3C_3\alpha_3' \\ &= 60^\circ - \angle A_3C_3\alpha_3' \\ &= \angle B_3C_3\alpha_3' - \angle A_3C_3\alpha_3'. \end{aligned}$$

Le triangle $C_3\alpha_3'\beta_3$ est la position du triangle $C_3B_3A_3$ après une rotation de 60° autour du point C_3. Si l'on imprime au segment rectiligne A_3B_3 une rotation de 60°, dans le même sens que la précédente, autour du point A_3, ce segment A_3B_3 devient équipollent à $\alpha_3'\beta_3$ et, de plus, coïncide avec le côté $A_3\gamma_3$ du triangle équilatéral $B_3\gamma_3A_3$. La figure $\alpha_3'\beta_3A_3\gamma_3$ est donc un parallélogramme et comme le sommet A coïncide avec le milieu du segment rectiligne $A_3\alpha_3'$,* les points β_3, A, γ_3 sont collinéaires et A est le milieu de $\beta_3\gamma_3$.

Par analogie, les sommets B, C sont les milieux des segments $\gamma_3\alpha_3$, $\alpha_3\beta_3$ et les points α_3, β_3, γ_3 coïncident avec les sommets A_2, B_2, C_2 du triangle obtenu en traçant les parallèles à BC, CA, AB par les sommets A, B, C. Un raisonnement analogue prouve que $\alpha_1' \equiv A_2$, $\beta_1' \equiv B_2$, $\gamma_1' \equiv C_2$.

2. Ce théorème conduit à une solution très rapide du problème suivant :

Construire un triangle ABC, connaissant les sommets A_1, B_1, C_1 (ou A_3, B_3, C_3), des triangles équilatéraux construits extérieurement (ou intérieurement) sur ses côtés.†

Les points A_1, B_1, C_1, par exemple, étant donnés, les sommets α_1', β_1', γ_1' des triangles équilatéraux construits intérieurement sur C_1B_1, B_1A_1, A_1C_1 déterminent les sommets $A_2 \equiv \alpha_1'$, $B_2 \equiv \beta_1'$, $C_2 \equiv \gamma_1'$ du triangle anticomplémentaire du triangle cherché ABC.

N.B.—Les points A_1, B_1, C_1 étant fixés, le problème est toujours possible.

Si θ_1 dèsigne l'angle de Brocard du triangle $A_1B_1C_1$, l'aire S_2 du triangle $\alpha_1'\beta_1'\gamma_1' \equiv A_2B_2C_2 = 4ABC = 4S$, a pour expression

$$S_2 = \tfrac{1}{2}S_1(5 - \sqrt{3}\cot\theta_1) = 4S,\ddagger$$

S_1 étant l'aire du triangle $A_1B_1C_1$.

Lorsque le triangle formé par les points donnés A_1, B_1, C_1 est tel que

$$\cot\theta_1 = 5/\sqrt{3}, \quad S_2 = 4S = 0.$$

Dans ce cas particulier, *les sommets*, A, B, C *du triangle cherché sont collinéaires ; la droite ainsi déterminée passe par le centre de gravité* G *commun au triangle* $A_1B_1C_1$ *et au triangle aplati* ABC.

* Van Aubel, *Nouvelle Correspondance*, question 607.

† E. Lemoine, *Nouvelles Annales de Mathématiques*, 1869, p. 40.

‡ Cette formule paraîtra prochainement dans un de nos *Suppléments à Mathesis*.

Les sommets A, B, C sont situés sur les cercles de Mackay du triangle $A_1B_1C_1$. La droite $\Delta \equiv (A, B, C)$, qui est l'axe d'homologie des triangles $(A_1B_1C_1, ABC)$, enveloppe la parabole de Kiepert du triangle $A_1B_1C_1$ dont la directrice est la droite d'Euler O_1G de ce triangle. L'orthopôle de la droite Δ, pour le triangle $A_1B_1C_1$, est situé sur la droite O_1G.

3. Considérons maintenant un quadrilatère convexe $ABCD$. Soient A', B', C', D' et E', F' les milieux des côtés AB, BC, CD, DA et des diagonales AC, BD qui se coupent en P ; A_1, B_1, C_1, D_1 et A_2, B_2, C_2, D_2 les sommets des triangles équilatéraux construits extérieurement puis intérieurement sur AB, BC, CD, DA. Soient aussi α, β, γ, δ les symétriques des sommets A, B, C, D par rapport aux points F', E', F', E' respectivement.

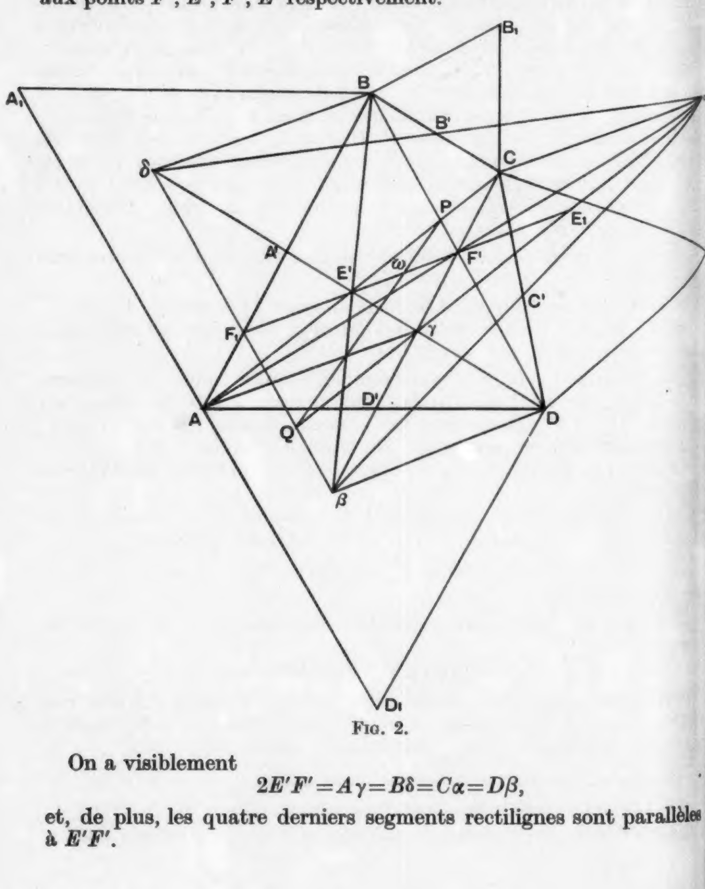


FIG. 2.

On a visiblement

$$2E'F' = A\gamma = B\delta = C\alpha = D\beta,$$

et, de plus, les quatre derniers segments rectilignes sont parallèles à $E'F'$.

Par suite, *les segments rectilignes* AC *et* $\alpha\gamma$, BD *et* $\beta\delta$ *sont équipollents ; les quadrilatères* BαCδ, CαDβ, DβAγ, AδBγ, BD$\beta\gamma$, AC$\alpha\gamma$ *sont des parallélogrammes.*

Les droites $\gamma\delta$, $\delta\alpha$, $\alpha\beta$, $\beta\gamma$ passent donc par les milieux A', B', C', D' des côtés du quadrilatère $ABCD$. On sait aussi que la barycentre des points A_1, B_1, C_1, D_1 (ou A_2, B_2, C_2, D_2) coïncide avec le barycentre Ω des points A, B, C, D * lequel est situé au milieu du segment $E'F'$. Les points E', F' sont les centres des parallélogrammes $AC\alpha\gamma$, $BD\beta\delta$, et si E_1, F_1 désignent les intersections de la droite $E'F'$ avec $\alpha\gamma$, $\beta\delta$, on a

$$F_1E' = E'F' = F'E_1.$$

Q étant l'intersection des droites $\alpha\gamma$, $\beta\delta$, les triangles QF_1E_1, $E'PF'$ se correspondent dans une homothétie (Ω, -3) puisque Ω est le milieu commun aux segments rectilignes $E'F'$ et F_1E_1. Les points, P, Ω, Q sont donc collinéaires, et

$$Q\Omega : P\Omega = -3. \quad \text{..............................(i)}$$

N.B.—Les aires algébriques des quadrilatères $ABCD$, $\alpha\beta\gamma\delta$ sont équivalentes car les diagonales AC, BD et $\alpha\gamma$, $\beta\delta$ sont des segments rectilignes équipollents. Les groupes de quatre points

$$(A, B, C, D),\ (A_1, B_1, C_1, D_1),\ (A_2, B_2, C_2, D_2),\ (\alpha, \beta, \gamma, \delta)$$

ont le même barycentre Ω.

Le centre de gravité de l'aire du quadrilatère $\alpha\beta\gamma\delta$ coïncide avec le point P de rencontre des diagonales du quadrilatère fondamental. Ceci résulte de la construction classique du centre de gravité de l'aire d'un quadrilatère et de la relation (i).

4. *Problème. Construire un quadrilatère convexe ABCD connaissant les sommets A_1, B_1, C_1, D_1 (ou A_2, B_2, C_2, D_2), des triangles équilatéraux construits extérieurement (ou intérieurement) sur ses côtés.*

Le point α, par exemple, qui est un des sommets du triangle anticomplémentaire du triangle ABD, coïncide donc, d'après notre théorème du début, avec le sommet du triangle équilatéral construit intérieurement sur le segment rectiligne D_1A_1. Par analogie, les points β, γ, δ sont déterminés par la connaissance des points B_1, C_1, D_1.

De plus, la droite E_1F_1, qui joint les milieux des segments $\alpha\gamma$, $\beta\delta$ contient les milieux E', F' des diagonales AC, BD du quadrilatère qu'il s'agit de construire, et ces points sont déterminés puisque

$$F_1E' = E'F' = F'E_1 = \tfrac{1}{3}E_1F_1.$$

Les sommets A, B, C, D du quadrilatère sont alors les symétriques des points α, β, γ, δ par rapport aux points F', E', F', E' respectivement.

N.B.—La construction du quadrilatère $ABCD$ connaissant les points A_2, B_2, C_2, D_2 se résout d'une manière identique, en vertu du théorème du premier paragraphe. Nous laissons au lecteur le soin de la discussion de ce problème. V. T.

* Propriété connue.

CONIC-TRACING IN LINE COORDINATES.

By S. P. Johnson.

Consider $$x : y : 1 = t^2 - 2t : t : t^2 - t - 1. \qquad \text{(i)}$$

The curve meets the line $lx + my + n = 0$ in points given by

$$l(t^2 - 2t) + mt + n(t^2 - t - 1) = 0.$$

For any given l, m, n there are two values of t. Hence the curve meets any line in two points and is therefore a conic. For a given l, m, n the values of t are the roots of the equation

$$t^2(l+n) + t(m - 2l - n) - n = 0.$$

These values of t are equal if the line is a tangent to the conic, and conversely. Hence the line equation of the conic is

$$4n(l+n) + (m - 2l - n)^2 = 0,$$

which reduces to

$$4l^2 + m^2 + 5n^2 - 4ml - 2mn + 8nl = 0. \qquad \text{(ii)}$$

The line equation may be taken as the medium of investigation of the dimensions of the curve, and its position relative to rectangular cartesian axes.

1. *The centre.*

The centre is the pole of the line at infinity, whose (homogeneous) line coordinates are (0, 0, 1). The pole of any line, say (l', m', n'), is the point

$$l' \frac{\partial f}{\partial l} + m' \frac{\partial f}{\partial m} + n' \frac{\partial f}{\partial n} = 0,$$

where $f(l, m, n)$ is the equation of the conic. Hence the centre of the conic (ii) is $\partial f / \partial n = 0$ or $4l - m + 5n = 0$. The centre, then, is the point whose coordinates in the corresponding point system are $(\frac{4}{5}, -\frac{1}{5})$.

2. *The equations of the axes.*

The equations of the axes, or their coordinates in the line system of coordinates, may be found by making use of the fact that they are the lines which join the foci; the one joining the real foci, and the other the conjugate imaginary foci.

For the foci, by the usual theory,

$$4l^2 + m^2 + 5n^2 - 4ml + 8nl - 2mn + \lambda(l^2 + m^2),$$

where λ is a constant, breaks up into two factors of the first degree; and the condition for this is

$$\begin{vmatrix} 4+\lambda, & -2, & 4 \\ -2\ \ , & 1+\lambda, & -1 \\ 4\ \ , & -1, & 5 \end{vmatrix} = 0,$$

which gives $\lambda = -2$ or $\frac{2}{5}$.

Corresponding to these values of λ we have for the equations of the foci

$$\phi \equiv 2l^2 - 4ml - m^2 - 2mn + 8nl + 5n^2 = 0 \quad \text{.................(iii)}$$

and $$\phi' \equiv 22l^2 - 20ml + 7m^2 + 40ln - 10mn + 25n^2 = 0, \quad \text{.........(iv)}$$

respectively. Now the coordinates of the line joining the two points whose equation is $\phi = 0$ are given by

$$\frac{\partial \phi}{\partial l} = 0, \quad \frac{\partial \phi}{\partial m} = 0, \quad \text{............................(v)}$$

or $$l - m + 2n = 0,$$

$$2l + m + n = 0,$$

that is, $$l : m : n = 1 : -1 : -1,$$

which, in the point system, is $x - y - 1 = 0$. Treating equation (iv) similarly, we ultimately obtain for the other axis

$$l : m : n = 5 : 5 : -3,$$

which, in the point system, is $5x + 5y - 3 = 0$.

3. *The lengths of the semi-axes.*

The distance between tangents to a conic that are parallel to an axis is equal to the length of the other axis.

Any line parallel to $(1, 1, -\frac{3}{5})$, the second axis just obtained, is $(1, 1, n)$. If this line is a tangent to the conic, its coordinates satisfy equation (ii). We therefore obtain for n the quadratic

$$5n^2 + 6n + 1 = 0, \quad \text{...............................(vi)}$$

the roots of which are -1 and $-\frac{1}{5}$. Hence the lines $(1, 1, -1)$ and $(1, 1, -\frac{1}{5})$ are the tangents to the conic at the ends of an axis of symmetry. These lines are at a distance $\frac{2}{5}\sqrt{2}$ apart. Thus the length of the semi-axis perpendicular to the axis $5x + 5y - 3 = 0$ is $\frac{1}{5}\sqrt{2}$.

For the other axis, proceeding similarly, we have the equation

$$5n^2 + 10n + 9 = 0, \quad \text{...........................(vii)}$$

the roots of which are clearly imaginary.

The conic is therefore a hyperbola, and taking the difference of the roots of equation (vii) and dropping $\sqrt{(-1)}$ we find that the length of the imaginary semi-axis is $\frac{1}{5}\sqrt{10}$.

4. *The eccentricity, foci and directrices.*

The lengths of the semi-axes being known, the eccentricity may be calculated by the usual formula connecting a and b in a hyperbola. For this hyperbola $e = \sqrt{6}$.

The foci have already been determined in equations (iii) and (iv). Factorising the left-hand member of these equations, the real and imaginary foci are found to be

$$5n + (4 + \sqrt{6})l - (1 - \sqrt{6})m = 0 \quad \text{and} \quad 5n + (4 - \sqrt{6})l - (1 + \sqrt{6})m = 0,$$

and $$5n+(4-\sqrt{-6})l-(1-\sqrt{-6})m=0$$

and $$5n+(4+\sqrt{-6})l-(1+\sqrt{-6})m=0,$$

respectively.

The directrices are the polars of the foci. If the coordinates of a directrix corresponding to one of these four foci, say,

$$5n+(4+\sqrt{6})l-(1-\sqrt{6})m=0,$$

be (λ, μ, ν) and the equation of the conic be written $f(l, m, n)=0$, the equation

$$l\frac{\partial f}{\partial \lambda}+m\frac{\partial f}{\partial \mu}+n\frac{\partial f}{\partial \nu}=0$$

is identifiable with

$$5n+(4+\sqrt{6})l-(1-\sqrt{6})m=0,$$

from which it appears that the directrix corresponding to this focus is $\{\sqrt{6}, \sqrt{6}, \frac{1}{5}(2+3\sqrt{6})\}$, or, in the point system,

$$5\sqrt{6}x+5\sqrt{6}y-2-3\sqrt{6}=0.$$

The other directrices may be found similarly.

Note 1. If the conic is an ellipse, both the equations such as (vi) and (vii) give real roots. Thus

$$x:y:1=2t+3:2t-3:2t^2+3,$$

which is an ellipse, gives $6n^2-4=0$ for the tangents parallel to the axis $(1, 1, 0)$ and $6n^2-12n=0$ for the tangents parallel to the axis $(1, -1, 1)$.

Note 2. Equations (v) are solvable in every case except when $\partial\phi/\partial l=0$, $\partial\phi/\partial m=0$ give equations of the form

$$\lambda_1(pl+qm+rn)=0,$$
$$\lambda_2(pl+qm+rn)=0,$$

as in the example in Note 1, for the value -2 of λ. Usually the axis may then be found from one of these two equations and $\partial\phi/\partial n=0$. But if $\partial\phi/\partial n$ leads to a third equation of the form

$$\lambda_3(pl+gm+rn)=0$$

the axis is then indeterminate. But it is easy to see that the line equation is that of a circle in such a case. For the centre of a circle is a double or singular focus. Hence the equation of the foci must be

$$(pl+qm+rn)^2=0,$$

and that of the circle

$$(pl+qm+rn)^2-k^2(l^2+m^2)=0, \quad \text{.................(viii)}$$

where k is a constant.

The centre is the point $pl+qm+rn=0$ and the length of the diameter is easily found by taking $(1, 0, n)$ as a typical tangent and proceeding as in section 3 above. It will be found that the diameter

of the circle whose equation is given in (viii) is $2k/r$. A typical circle is

$$x : y : 1 = 2 : (t+1)^2 : t^2+1.$$

The parabola. If the conic is a parabola, the work is considerably shortened, though the general principle is the same. Consider

$$x : y : 1 = t^2+1 : t^2+2t+1 : 1.$$

The line equation is

$$l^2 + 2lm + nl + mn = 0. \qquad \text{(ix)}$$

The foci. $l^2 + 2lm + nl + mn + \lambda(l^2 + m^2)$ breaks up into first degree factors if $\lambda = \frac{1}{2}$, and we have

$$(l+m)(3l+m+2n) = 0.$$

The finite focus is $(\frac{3}{2}, \frac{1}{2})$ and the other focus is the point whose equation is $l+m=0$, which is a point at infinity on the line $x=y$; this line is, accordingly, parallel to the axis of the parabola.

The axis. As in 2 above, we get the equations

$$3l+2m+n=0, \quad 2l+m+n=0,$$

from which $l : m : n = 1 : -1 : -1$. Hence the axis is $x-y-1=0$.

The tangent at the vertex. Any line perpendicular to the axis has coordinates $(1, 1, n)$. This line touches the parabola if its coordinates satisfy (ix), when $n = -\frac{3}{2}$. Hence the equation of the tangent at the vertex is $2x+2y-3=0$. The latus rectum is four times the distance of the focus from the tangent at the vertex, and is $\sqrt{2}$.

Note 3. By considering the nature of the tangents drawn to the conic from the pole of the line at infinity, which are the asymptotes and are real, coincident or imaginary according as the conic is a hyperbola, parabola or ellipse, it is easily established that

$$al^2 + bm^2 + cn^2 + 2fmn + 2gnl + 2hlm = 0$$

is an ellipse, hyperbola or parabola according as $c\Delta$ is positive, negative or zero, where Δ stands as usual for the determinant

$$\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix}.$$

Since Δ cannot vanish unless the conic is two points—a degenerate case which is nugatory for the purpose of the present discussion—a necessary and sufficient condition for a parabola is $c=0$. If when c is present in the equation it be written with a positive sign, the necessary and sufficient condition for each type of conic may be more concisely stated:

for an ellipse, $\Delta > 0$;

for a hyperbola, $\Delta < 0$;

for a parabola, $c = 0$.

S. P. J.

THE HARMONIC CONICS.

By J. Clemow.

The following notes are the outcome of a course of lectures to pupils on invariants. The methods are new to me, but I publish them with hesitation and apologise beforehand should they prove well known.

Notation.

$$S \equiv (abcfgh \between xyz)^2, \quad \Sigma \equiv (ABCFGH \between XYZ)^2.$$

P_1 denotes the point (x_1, y_1, z_1), p the line $Xx + Yy + Zz = 0$, so that the tangential coordinates of p are (X, Y, Z).

$$\xi \equiv \tfrac{1}{2}\frac{\partial S}{\partial x}, \quad \eta \equiv \tfrac{1}{2}\frac{\partial S}{\partial y}, \quad \zeta \equiv \tfrac{1}{2}\frac{\partial S}{\partial z}.$$

$$S_1 \equiv \xi_1 x + \eta_1 y + \zeta_1 z \equiv \xi x_1 + \eta y_1 + \zeta z_1 = 0$$

is the pole of P_1 with respect to S.

$$S_{11} \equiv (abcfgh \between x_1 y_1 z_1)^2,$$

and so on in a similar manner for tangential forms.

The Harmonic Envelope.

Any point on the line P_1P_2 is of the form $\kappa_1 P_1 + \kappa_2 P_2$, where κ_1, κ_2 are parameters. This point lies on the conic $S = 0$ if

$$\kappa_1^2 S_{11} + 2\kappa_1\kappa_2 S_{12} + \kappa_2^2 S_{22} = 0, \quad \text{.....................(i)}$$

and on conic $S' = 0$ if

$$\kappa_1^2 S_{11}' + 2\kappa_1\kappa_2 S_{12}' + \kappa_2^2 S_{22}' = 0. \quad \text{.....................(ii)}$$

Equations (i) and (ii) may be regarded as giving κ_1/κ_2 for the points A, B; A', B' in which P_1P_2 meets S, S', and hence

$$(AB, A'B') = -1$$

if the roots of (i) harmonically separate those of (ii), that is, if

$$S_{11}S_{22}' - 2S_{12}S_{12}' + S_{22}S_{11}' = 0.$$

Thus allowing P_2 to vary, we get

$$S_{11}S' - 2S_1S_1' + SS_{11}' = 0, \quad \text{.......................(iii)}$$

an equation of the second degree representing the two lines that can be drawn through P_1 meeting S, S' in pairs of harmonically separated points.

Hence: ***a line which meets one conic in a pair of points harmonically separated by its intersections with another conic, touches a conic.***

From (iii) two results follow immediately. The tangents from P_1 to the harmonic envelope Φ of S and S' are given by (iii). They are perpendicular if

$$(a' + b')S_{11} - 2\xi_1\xi_1' - 2\eta_1\eta_1' + (a+b)S_{11}' = 0.$$

If S and S' are rectangular hyperbolas this reduces to

$$\xi_1\xi_1' + \eta_1\eta_1' = 0.$$

Hence: *the director circle of the harmonic envelope of two rectangular hyperbolas is such that the polars of any point of it with respect to the hyperbolas are perpendicular.*

Again, if S' is a coincident line pair, $S' \equiv p^2$, say, then (iii) becomes

$$S_{11}p^2 - 2S_1p_1p + Sp_1^2 = 0. \qquad \text{.....................(iv)}$$

The equation (iv) represents therefore the pair of lines joining P_1 to the intersections of p with S.

Covariants.

The methods used in obtaining the following results are longer than those usually employed, but they have the merit that they do not depend at all on a previous knowledge of the results.

The operation of taking the tangential form of a point equation, or *vice versa*, we denote by a bar; thus $\overline{S} \equiv \Sigma$, $\overline{\Sigma} \equiv \Delta S$. The operation of forming the F-conic of two tangential conics Σ, Σ' we denote by brackets, thus:

$$\{\Sigma, \Sigma'\} \equiv \mathrm{F}.$$

Similarly, $$\{S, S'\} \equiv \Phi.$$

Lemma. $$\{\Sigma, \Phi\} \equiv \Theta S + \Delta S'.$$

Refer S, S' to their common self-conjugate triangle:

$$S \equiv ax^2 + by^2 + cz^2 = 0;$$
$$S' \equiv a'x^2 + b'y^2 + c'z^2 = 0.$$

Then $$\Theta = a'bc + b'ca + c'ab, \quad \Delta = abc;$$
$$\Theta' = ab'c' + bc'a' + ca'b', \quad \Delta' = a'b'c'.$$

Then $$\Sigma \equiv bcX^2 + caY^2 + abZ^2 = 0;$$

thus $$\mathrm{F} = \{\Sigma, \Sigma'\} \equiv aa'(cb' + c'b)x^2 + \ldots + \ldots = 0,$$
$$\Phi = \{S, S'\} \equiv (bc' + b'c)X^2 + \ldots + \ldots = 0.$$

Hence $$\{\Sigma, \Phi\} \equiv \{ca(ab' + a'b)\} + ab(c'a + ca')\}x^2 + \ldots + \ldots$$
$$= \{a(a'bc + b'ca + c'ab) + a'abc\}x^2 + \ldots + \ldots$$
$$= \Theta S + \Delta S'.$$

Similarly $$\{S, \mathrm{F}\} \equiv \Theta'\Sigma + \Delta\Sigma'.$$

The first problem is to find $\overline{\Phi}$, the point equation of the harmonic envelope. Now we know that

$$\overline{\lambda S + \lambda' S'} \equiv \lambda^2\Sigma + \lambda\lambda'\Phi + \lambda'^2\Sigma'. \qquad \text{.....................(v)}$$

From the nature of the transformation it is clear that, apart from a constant factor, $\overline{\overline{\lambda S + \lambda' S'}}$ must be identically $\lambda S + \lambda' S'$. We proceed to find $\overline{\lambda^2\Sigma + \lambda\lambda'\Phi + \lambda'^2\Sigma'}$. Writing

$$\Phi \equiv (\mathfrak{A}\mathfrak{B}\mathfrak{C}\mathfrak{F}\mathfrak{G}\mathfrak{H} \between XYZ)^2,$$

the discriminant of (v) is

$$\begin{vmatrix} \lambda^2 A + \lambda\lambda'\mathfrak{A} + \lambda'^2 A' & & \\ & \lambda^2 B + \lambda\lambda'\mathfrak{B} + \lambda'^2 B', & \\ & & \lambda^2 C + \lambda\lambda'\mathfrak{C} + \lambda'^2 C' \end{vmatrix}.$$

Taking the minors of this, we see that

$$\overline{\lambda^2\Sigma + \lambda\lambda'\Phi + \lambda'^2\Sigma'} \equiv \lambda^4\overline{\Sigma} + \lambda^2\lambda'^2(\overline{\Phi} + \mathbf{F}) + \lambda'^4\overline{\Sigma}'$$
$$+ \lambda^3\lambda'\{\Sigma, \Phi\} + \lambda\lambda'^3\{\Sigma', \Phi\}.$$

But this by the lemma is

$$\lambda^4\Delta S + \lambda^2\lambda'^2(\overline{\Phi} + \mathbf{F}) + \lambda'^4 S'\Delta'$$
$$+ \lambda^3\lambda'(\Theta S + \Delta S') + \lambda\lambda'^3(\Theta' S' + \Delta' S)$$

or $$(\lambda S + \lambda' S')(\Delta\lambda^3 + \Theta\lambda^2\lambda' + \Theta'\lambda\lambda'^2 + \Delta\lambda'^3)$$
$$+ \lambda^2\lambda'^2(\overline{\Phi} + \mathbf{F} - \Theta S' - \Theta' S).$$

But this must be precisely $\lambda S + \lambda' S'$ except for a factor; thus the coefficient of $\lambda^2\lambda'^2$ must vanish, and so

$$\overline{\Phi} \equiv \Theta S' + \Theta' S - \mathbf{F}, \quad \text{(vi)}$$

which is the required result.

The next problem is to determine the equations of the reciprocal of S with respect to S'. Let $\mathfrak{S}$ be its point equation; then at once from the geometry

$$\mathfrak{S} \equiv \lambda' S' + \lambda\mathbf{F} = 0, \quad \text{(vii)}$$

$$\overline{\mathfrak{S}} \equiv \mu'\Sigma' + \mu\Phi = 0, \quad \text{(viii)}$$

where λ, λ', μ, μ' are constants to be determined.

Now $$\overline{\overline{\mathfrak{S}}} \equiv \overline{\mu'\Sigma' + \mu\Phi}$$
$$\equiv \mu'^2\overline{\Sigma}' + \mu\mu'\{\Sigma', \Phi\} + \mu^2\overline{\Phi}.$$

But by the lemma and (vi) this is

$$\mu'^2 S'\Delta' + \mu\mu'(\Theta S' + \Delta' S) + \mu^2(\Theta S' + \Theta' S - \mathbf{F})$$
$$= S'(\Delta'\mu'^2 + \Theta'\mu\mu' + \Theta\mu^2) + S(\mu\mu'\Delta' + \Theta'\mu^2) - \mathbf{F}\mu^2. \quad \text{(ix)}$$

But $\overline{\overline{\mathfrak{S}}}$ must be precisely $\mathfrak{S}$ except for a possible factor. Thus (ix) must be of the form of (vii), and hence

$$\mu\mu'\Delta' + \Theta'\mu^2 = 0.$$

Now $\mu \neq 0$; thus $\mu'/\Theta' = -\mu/\Delta'$, and so by (viii)

$$\overline{\mathfrak{S}} \equiv \Theta\Sigma' - \Delta'\Phi. \quad \text{(x)}$$

Again, from (ix),

$$\overline{\overline{\mathfrak{S}}} \equiv S'(\Delta'\Theta'^2 - \Theta'^2\Delta' + \Theta\Delta'^2) - \Delta'^2\mathbf{F}$$
$$\equiv \Delta'^2(\Theta S' - \mathbf{F}).$$

Thus $$\mathfrak{S} \equiv \Theta S' - \mathbf{F}. \quad \text{(xi)}$$

A final example of the method. Let S^1, S^2 be two conics of the pencil defined by S and S'; $S^1 \equiv S + \lambda_1 S'$, $S^2 = S + \lambda_2 S'$.

To find the harmonic envelope $\{S^1, S^2\}$, consider

$$S^1 + \mu S^2 \equiv (1+\mu) S + (\lambda_1 + \mu\lambda_2) S'.$$

Transforming this to tangentials,

$$\Sigma^1 + \mu\{S^1, S^2\} + \mu^2\Sigma^2 \equiv (1+\mu)^2\Sigma + (\lambda_1 + \mu\lambda_2)^2\Sigma' + (1+\mu)(\lambda_1+\mu\lambda_2)\Phi.$$

Equating coefficients of μ, we have

$$\{S^1, S^2\} \equiv 2\Sigma + (\lambda_1+\lambda_2)\Phi + 2\lambda_1\lambda_2\Sigma'. \quad \text{..............(xii)}$$

Converting this to point coordinates, we obtain

$$\overline{\{S^1, S^2\}} \equiv 4\Delta S + (\lambda_1+\lambda_2)^2\overline{\Phi} + 4\lambda_1^2\lambda_2^2\Delta' S' + 2(\lambda_1+\lambda_2)\{\Phi, \Sigma\} + 2\lambda_1\lambda_2(\lambda_1+\lambda_2)\{\Phi, \Sigma'\} + 4\lambda_1\lambda_2\mathbf{F}. \quad \text{.................(xiii)}$$

Again, $\Sigma^1 + \mu\Sigma^2 \equiv (\Sigma + \lambda_1\Phi + \lambda_1^2\Sigma') + \mu(\Sigma + \lambda_2\Phi + \lambda_2^2\Sigma')$

$$\equiv (1+\mu)\Sigma + (\lambda_1+\mu\lambda_2)\Phi + (\lambda_1^2 + \mu\lambda_2^2)\Sigma'.$$

Thus $\overline{\Sigma^1 + \mu\Sigma^2} \equiv (1+\mu)^2\Delta S + (\lambda_1 + \mu\lambda_2^2)\overline{\Phi} + (\lambda_1^2 + \mu\lambda_2^2)\Delta' S'$

$$+ (1+\mu)(\lambda_1+\mu\lambda_2)\{\Phi, \Sigma\}$$

$$+ (\lambda_1+\mu\lambda_2)(\lambda_1^2+\mu\lambda_2)^2\{\Phi, \Sigma'\} + (1+\mu)(\lambda_1^2+\mu\lambda_2^2)^2\mathbf{F}.$$

But $\overline{\Sigma^1 + \mu\Sigma^2} \equiv S^1\Delta^1 + \mu\{\Sigma^1, \Sigma^2\} + \mu^2 S^2\Delta^2$.

Equating coefficients of μ, we have

$$\{\Sigma^1, \Sigma^2\} \equiv 2\Delta S + 2\lambda_1\lambda_2\overline{\Phi} + 2\lambda_1^2\lambda_2^2 S'\Delta' + (\lambda_1+\lambda_2)\{\Sigma, \Phi\} + \lambda_1\lambda_2(\lambda_1+\lambda_2)\{\Sigma', \Phi\} + (\lambda_1^2+\lambda_2^2)\mathbf{F}. \quad \text{.........(xiv)}$$

Thus from (xiii) and (xiv) we have immediately

$$\overline{\{S^1, S^2\}} - 2\{\Sigma^1, \Sigma^2\} \equiv (\lambda_1-\lambda_2)^2\overline{\Phi} - 2(\lambda_1-\lambda_2)^2\mathbf{F}.$$

Thus the harmonic curves of S^1, S^2, any two curves of the pencil $S + \lambda S'$, always intersect on the conic $\overline{\Phi} - 2\mathbf{F}$, that is, on

$$\Theta S' + \Theta' S - 3\mathbf{F}.$$

Hence: *the locus of intersections of the harmonic locus and the harmonic envelope of two conics varying in a pencil is a conic.*

Other covariant forms can be obtained quite straightforwardly, but they are rather artificial and of no particular geometrical interest.

I have to thank my late pupils, Messrs. R. L. Brooks and A. G. Matthewman, of Hymer's College, for their ideas and suggestions.

J. C.

1006. A Numerical Curiosity.

When e, a, t are the digits 2, 4, 3,

$$\frac{eat}{tea} = \frac{tea}{ate}.$$

—[Per Mr. N. Anning.]

THE ORTHOCENTRIC SIMPLEX IN SPACE OF THREE AND HIGHER DIMENSIONS.*

By H. Lob.

By a simplex is meant the figure in general flat space of which the triangle and tetrahedron are examples. In space of n dimensions the simplex has $(n+1)$ vertices and $(n+1)$ faces, each face consisting of a prime, *i.e.* a space of $(n-1)$ dimensions, containing n of the vertices and hence itself a simplex in $(n-1)$-dimensional space. There is a general likeness between the triangle and simplexes of higher order, but this is marred by the fact that, in general, a simplex does not possess an orthocentre.

It is, however, possible for a simplex in general space to have an orthocentre, and in such a case the simplex presents a very complete analogy to the triangle. The purpose of the present slight paper is to indicate how the resemblance to the triangle can be worked out simply by means of elementary vector algebra. (*Note.* Throughout this paper, the scalar product of vectors α, β is written $\alpha\beta$. No vector products occur.)

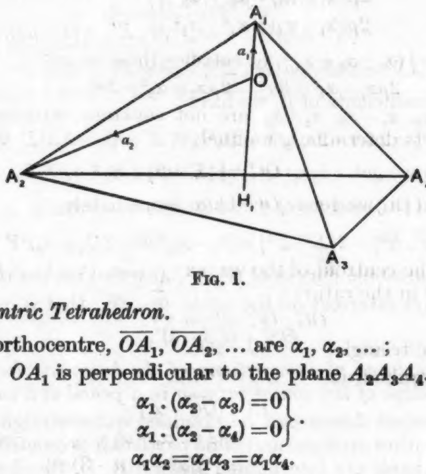


Fig. 1.

The Orthocentric Tetrahedron.

O is the orthocentre, $\overline{OA_1}$, $\overline{OA_2}$, ... are α_1, α_2, ...;

OA_1 is perpendicular to the plane $A_2A_3A_4$.

Then $$\left.\begin{aligned}\alpha_1\,.\,(\alpha_2-\alpha_3)&=0\\ \alpha_1\,.\,(\alpha_2-\alpha_4)&=0\end{aligned}\right\}.$$

Thus $$\alpha_1\alpha_2=\alpha_1\alpha_3=\alpha_1\alpha_4.$$

Similarly $$\alpha_2\alpha_1=\alpha_2\alpha_3=\alpha_2\alpha_4,$$
$$\alpha_3\alpha_1=\alpha_3\alpha_4.$$

Thus all the scalar products are equal ($=\sigma^2$, say).

This is merely stating that

$$OA_1\,.\,OH_1=\sigma^2,$$

so that σ^2 is the square of the radius of the polar sphere.

* A paper read at the Annual Meeting of the Mathematical Association, 8th January, 1935.

Similarly, in the general simplex,

$$\alpha_r\alpha_s=\sigma^2.$$

Obvious properties.

(1) Any two edges which do not meet are perpendicular; for example,

$$A_1A_2 \text{ is perpendicular to } A_3A_4.$$

For $\qquad \overline{A_1A_2}=\alpha_2-\alpha_1$ and $\overline{A_3A_4}=\alpha_4-\alpha_3$,

and
$$(\alpha_2-\alpha_1)(\alpha_4-\alpha_3)=\sigma^2-\sigma^2-\sigma^2+\sigma^2$$
$$=0.$$

(2) Each face is itself an orthocentric simplex in $[n-1]$, with the foot of the perpendicular from the opposite vertex as its orthocentre.

Circumsphere.

If C is the circumcentre and $\overline{OC}=\rho$,

then $\qquad (\rho-\alpha_1)^2=(\rho-\alpha_2)^2=(\rho-\alpha_3)^2=(\rho-\alpha_4)^2=R^2.$

Therefore $\qquad \rho^2-2\rho\alpha_1+\alpha_1{}^2=\rho^2-2\rho\alpha_2+\alpha_2{}^2=\ldots=R^2.$

Hence
$$\left.\begin{aligned}2\rho(\alpha_1-\alpha_2)&=\alpha_1{}^2-\alpha_2{}^2\\2\rho(\alpha_1-\alpha_3)&=\alpha_1{}^2-\alpha_3{}^2\\2\rho(\alpha_1-\alpha_4)&=\alpha_1{}^2-\alpha_4{}^2\end{aligned}\right\}.$$

Clearly $\rho=\frac{1}{2}(\alpha_1+\alpha_2+\alpha_3+\alpha_4)$ satisfies these equations,

for $\qquad 2\rho\alpha_1=\alpha_1{}^2+3\sigma^2, \quad 2\rho\alpha_2=\alpha_2{}^2+3\sigma^2, \ldots .$

Also $\alpha_1-\alpha_2$, $\alpha_1-\alpha_3$, $\alpha_1-\alpha_4$ are not coplanar, so that the three scalar products determine ρ uniquely.

Thus in [3] we get $\qquad \overline{OC}=\frac{1}{2}(\alpha_1+\alpha_2+\alpha_3+\alpha_4).$

Similarly in $[n]$ we get $\qquad \overline{OC}=\frac{1}{2}(\alpha_1+\alpha_2+\ldots+\alpha_{n+1}).$

Corollaries.

(*a*) Since the centroid of the vertices is $(\Sigma\alpha)/(n+1)$, it follows that G divides CO in the ratio

$$OG:OC=2:n+1.$$

Thus in the triangle, $\qquad OG=\frac{2}{3}OC,$

and in the tetrahedron, $\qquad OG=\frac{1}{2}OC.$

(*b*)
$$\begin{aligned}R^2&=(\overline{OC}-\alpha_1)^2\\&=\{\tfrac{1}{2}(-\alpha_1+\alpha_2+\alpha_3+\ldots+\alpha_{n+1})\}^2\\&=\tfrac{1}{4}\{\Sigma\alpha^2+n(n-3)\sigma^2\}.\end{aligned}$$

Thus in the triangle, $\qquad R^2=\frac{1}{4}(\Sigma\alpha^2-2\sigma^2),$

and in the tetrahedron, $\qquad R^2=\frac{1}{4}\Sigma\alpha^2.$

(*c*)
$$\begin{aligned}OC^2&=\tfrac{1}{4}(\alpha_1+\alpha_2+\ldots+\alpha_{n+1})^2\\&=\tfrac{1}{4}\{\Sigma\alpha^2+n(n+1)\sigma^2\}\\&=R^2+n\sigma^2.\end{aligned}$$

(*d*) The simplex has an Euler Line, and the Euler Line of the simplex meets the Euler Lines of the faces.

Analogues to the Nine-Point Circle.

(1) The mid-points of the edges lie on a sphere with the mid-point of $\overline{OC}$ as centre.

For if M is the mid-point of OC and K the mid-point of any edge, A_2A_4 say, then

$$\overline{MK} = \tfrac{1}{2}(\alpha_2 + \alpha_4) - \tfrac{1}{4}(\alpha_1 + \alpha_2 + \ldots + \alpha_{n+1}).$$

Therefore
$$MK^2 = \{\Sigma\alpha^2 + (n^2 - 7n + 8)\sigma^2\}/16$$
$$= \tfrac{1}{4}\{R^2 - (n-2)\sigma^2\}.$$

Thus in the triangle, the square of the radius is $\tfrac{1}{4}R^2$.

(2) The feet of the perpendiculars from the vertices to the opposite faces lie on a sphere of radius R/n and centre $1/n$th of the distance along OC.

Consider the section of the circumsphere by the plane AOC, where A is any vertex. Let H be the foot of the perpendicular from A to the opposite face.

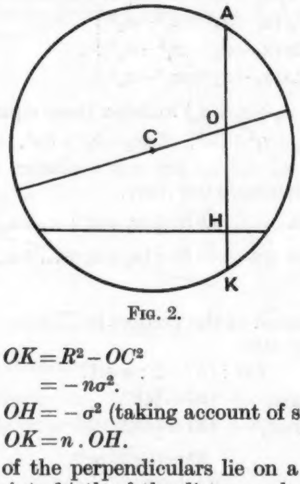


Fig. 2.

$$AO \,.\, OK = R^2 - OC^2$$
$$= -n\sigma^2.$$

Also $\quad AO \,.\, OH = -\sigma^2$ (taking account of sense of segments).

Thus $\quad OK = n \,.\, OH.$

Hence the feet of the perpendiculars lie on a sphere which cuts OA_1, OA_2, ... at points $1/n$th of the distance along them, the pedal sphere.

(3) The mid-points of OA_1, OA_2, ... lie on a sphere of radius $\tfrac{1}{2}R$.

The Pedal Sphere goes through the Centroids of the Faces.

If G_1 is the centroid of the face opposite A_1,

then
$$\overline{OG_1} = (\alpha_2 + \alpha_3 + \ldots + \alpha_{n+1})/n.$$

If P is the centre of the pedal sphere,
$$\overline{OP} = (\alpha_1 + \alpha_2 + \ldots + \alpha_{n+1})/2n.$$

Thus $$\overline{PG_1}=(-\alpha_1+\alpha_2+\alpha_3+\ldots+\alpha_{n+1})/2n$$
$$=\overline{CA_1}/n.$$

Hence the length $PG_1=R/n$.

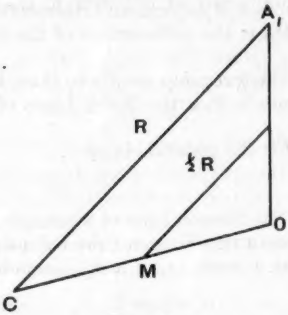


FIG. 3.

The Pedal, Mid-point, Circum- and Polar Spheres are Coaxial.

If σ is the vector to a common point of the circumsphere and the polar sphere, then

$$(\sigma-\overline{OC})^2=R^2.$$

Therefore $\sigma^2-2\sigma\,.\,\overline{OC}+OC^2=R^2.$

But $$OC^2=R^2+n\sigma^2.$$

Thus $$2\sigma\,.\,\overline{OC}=(n+1)\sigma^2,$$

giving the equation of the common prime.

In just the same way we can find the common prime of the polar sphere and pedal sphere, and of the polar sphere and mid-point sphere. The equation of each is

$$2\sigma\,.\,\overline{OC}=(n+1)\sigma^2.$$

Inscribed Sphere.

For dealing with the inscribed sphere, we make use of the following principle:

If $(a_1a_2\ldots)$ are the polyhedral coordinates of a point P referred to the simplex, then the vector

$$\overline{OP}=a_1\alpha_1+a_2\alpha_2+\ldots+a_{n+1}\alpha_{n+1}.$$

This follows at once from the fact that P is the centroid of masses $a_1, a_2, \ldots$ at the vertices of the simplex.

Now the coordinates of I, the incentre, are $\left(\frac{r}{h_1}, \frac{r}{h_2}, \ldots\right)$, where $h_1, h_2, \ldots$ are the heights of the simplex.

And since the sum of the coordinates is 1, we have

$$r\left(\frac{1}{h_1}+\frac{1}{h_2}+\ldots\right)=1.$$

From this we get

$$OI^2 = nr^2 + \sigma^2,$$
$$IC^2 = nr^2 + R^2 - r\Sigma OA.$$

The five points $A_1A_2A_3A_4O$ form an orthocentric system, of which any one of the points is the orthocentre of the tetrahedron formed by the other four.

We can apply all the foregoing results to these tetrahedra, and one property that emerges is that the Euler Lines of all the tetrahedra are concurrent.

This is true also for the general simplex.

Simson Plane.

Corresponding to the Simson Line of a triangle, we have a Simson Plane of a tetrahedron and a Simson Prime of a simplex.

The condition that a point $(x_1x_2 \dots x_{n+1})$ should possess a Simson Prime is $\dfrac{1}{x_1h_1^2} + \dfrac{1}{x_2h_2^2} + \dots = 0$, where h_1, h_2, ... are the heights of the simplex.

There is an interesting property of the triangle that if X is a point having a Simson Line, then OX is bisected by this Simson Line. In the case of the orthocentric simplex, OX is divided in the ratio of $n-1:1$.

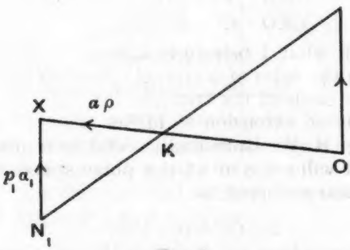


FIG. 4.

Let N_1, N_2, ... be the feet of the perpendiculars from X to the faces, and let A_1N_1 meet OX in K.

Let $\overline{OX} = \rho$ and $\overline{KX} = a\rho$.

Then $\overline{N_1X}$ is parallel to α_1 and so equals $p_1\alpha_1$, say.

$\overline{N_2X}$ is parallel to α_2 and so equals $p_2\alpha_2$,

....

Thus $\overline{KN_1} = a\rho - p_1\alpha_1$, $\overline{KN_2} = a\rho - p_2\alpha_2$, ...

Now KN_1, KN_2, ... all lie in the Simson Prime of X.

Hence if τ is a vector perpendicular to this prime, we have

$$(a\rho - p_1\alpha_1)\tau = 0 = (a\rho - p_2\alpha_2)\tau = \dots .$$

Therefore $$a\rho\tau = p_1\alpha_1\tau = p_2\alpha_2\tau = \dots .$$

Also, $$\rho = x_1\alpha_1 + x_2\alpha_2 + \dots ;$$

thus
$$\rho\tau = x_1\alpha_1\tau + x_2\alpha_2\tau + \dots$$
$$= a\rho\tau\left(\frac{x_1}{p_1} + \frac{x_2}{p_2} + \dots\right).$$

Now $$x_1 = \frac{N_1X}{h_1} \quad \text{and} \quad p_1 = \frac{N_1X}{OA_1}.$$

Hence $$\frac{x_1}{p_1} = \frac{OA_1}{h_1},$$

and similarly $$\frac{x_2}{p_2} = \frac{OA_2}{h_2},$$

$$\dots\dots\dots\dots$$

Thus we get
$$\frac{1}{a} = \Sigma\frac{x_1}{p_1}$$
$$= \Sigma\frac{OA_1}{h_1}$$
$$= n + 1 - (\text{sum of coordinates of } O)$$
$$= n + 1 - 1$$
$$= n.$$

Hence $$\frac{XK}{XO} = \frac{1}{n}.$$

I conclude with what I believe is known as Gaskin's Theorem : if a conic touches the sides of a triangle, its director circle is orthogonal to the polar circle of the triangle.

The following is an extension to higher space of a method communicated by Mr. H. W. Richmond. All I have done is to Bowdlerize it, so that it will come into the n-dimensional geometry.

Let σ_1 be the polar radius of the face opposite A_1.

Then $$\sigma_1^2 + H_1O^2 = \sigma^2.$$

This follows at once from the fact that the polar sphere of the face is the section of the polar sphere of the simplex and H_1 is its centre.

Now if the faces through A_1 are mutually perpendicular, we have O coinciding with A_1, $\sigma^2 = 0$, and thus $\sigma_1^2 = -H_1A_1^2$.

Take a quadric touching the mutually perpendicular faces, and let C be its centre and $a_1, a_2, \dots a_n$ be its semi-axes.

Then A_1 lies on the director sphere.

Hence $$CA_1^2 = a_1^2 + a_2^2 + \dots + a_n^2.$$

A limiting case of such a quadric is the " disc " lying in the face opposite A_1 and with one of its semi-axes (say a_n) $= 0$.

For this " disc " we have
$$a_1^2 + a_2^2 + \dots + a_{n-1}^2 = CA_1^2$$
$$= C_1H_1^2 + H_1A_1^2$$
$$= C_1H_1^2 - \sigma_1^2.$$

Thus $$\Sigma a_1^2 + \sigma_1^2 = C_1H_1^2.$$

But this disc is really a quadric inscribed in the simplex formed by the face, and Σa_1^2 is the square of the radius of its director sphere, and so the director sphere is orthogonal to the polar sphere of the face.

To complete the proof one ought to shew that, starting with any simplex in space of $(n-1)$ dimensions, we can form a simplex of the required type having this as a face. There is no difficulty here, but I omit the proof.

As corollaries in space of three dimensions, the following may be mentioned :

(*a*) If a paraboloid touches the faces of an orthocentric tetrahedron, its director plane passes through the orthocentre.

(*b*) This reciprocates into :

If a rectangular hyperboloid (having an asymptotic cone with three mutually perpendicular generators) passes through the four vertices, it also goes through the orthocentre.

(*c*) If *any* tetrahedron is self-polar with respect to a quadric Σ, then the circumsphere S is orthogonal to the director sphere of Σ.

For a tetrahedron can be described touching Σ and self-polar to S (and therefore orthocentric) ; hence the director sphere of Σ is orthogonal to S.

The President regretted that there was no time available for comments or questions on the paper, which suggested an attractive topic for the Junior Mathematical Association.

1007. Solutions of double acrostics : Gleaning 945 (XVII, p. 264 ; October, 1933) ; Gleaning 989 (XVIII, p. 254 ; October, 1934).

M	aximu	M
A	r	E
T	riden	T
H	yperbol	A
E	mir	P
M	arc	H
A	ntinom	Y
T	ripo	S
I	O O	I
C	ubi	C
S	olid	S

P	arse	C
E	ch	O
R	irro	M
M	ool	B
m U	ltipl	I cand
T	homso	N
A	lgebr	A
T	ai	T
I	naud	I
O	ccultati	O n
N	ewto	N
S	ophu	S Lie

THE FIRST ENCOUNTER WITH A LIMIT.*

The President explained that the genesis of the discussion about to take place on " The First Encounter with a Limit " was in certain references to limits which appeared in the *Algebra Report*. In the latter there were one or two remarks which upset some sensitive folk. Although there was ample precedent for discussing at length a report which had appeared during the year, it seemed best not to have on this occasion a discussion rambling over the whole of the *Algebra Report*, but to confine attention to the matter of the first encounter with a limit. The subject arose historically in that way, but it was not in the least intended that it should be confined to criticism of the points of view expressed in the *Report*. Those whose names were on the agenda had a completely free hand in their treatment of the matter, as, of course, had those who wished to make comments as the discussion developed.

Mr. T. A. A. Broadbent (University of Reading): The teaching of calculus takes an increasingly prominent part in the school curriculum of our times, and one inevitable consequence is that the idea of a limit is much more important than ever before. Not only must more emphasis be laid on the concept, but in the actual teaching of it we have to grapple very closely with the really difficult problem of combining logic and attractive presentation in due proportion to make the mixture both palatable and valuable—we must analyse the idea without spoiling its intrinsic charm.

The first encounter with a limit marks the dividing line between the elementary and the advanced parts of a school course. Here we have not a new manipulation of old operations, but a new operation; not a new trick, but a new idea. There is still a feeling that the limit must first be met in connection with the geometric series; some of this feeling is a survival from the days when most of the small amount of calculus teaching given in schools was on the manipulative side, and the limit which a schoolboy then encountered was really little more than another trick to be acquired in the progress towards the rubber of a University scholarship. Nowadays the connection between the series and the idea of a limit is relaxed, but there may still be appropriateness in the discussion of the subject by an Association whose earliest title contained the wild word *Reform*.

If in this investigation we are to seek for sound general principles, there are two which offer guidance. One is that, in teaching an elementary piece of work, the teacher should always have as his background, or say, in the back of his mind, that higher discipline of which the point with which he is concerned is an elementary manifestation; we do not teach small boys non-euclidean geometry, but our own handling of the parallel postulate will be surer if we have some recollection of the existence of Lobachewsky and Rie-

* A discussion at the Annual Meeting of the Mathematical Association, 8th January, 1935. *See also* letters from Mr. F. C. Boon and Mr. N. M. Gibbins, printed on pp. 131-4 of this issue of the *Gazette*.

mann. And, secondly, we must above all be simple, we must use plausible and understandable arguments, but only if they are such that when the time comes for the rigorous presentation, the arguments have merely to be strengthened, not utterly condemned and abandoned ; useless lies are really wicked.

Now how does the encounter through the geometric series answer these requirements ? May I here interpolate a defence against any suggestion that I am prejudiced against infinite series ; their domain is to me perhaps the most delightful in all mathematics, but one may be a member of a White Rose society without quite approving of the repeated appearance of King Charles' head in Mr. Dick's memorial. To encounter the limit through the series is to violate the first of our two principles, for it is the sequence which is important, the sequence which must form the basis of more advanced work, for infinite series, infinite products and other infinite processes are simply special cases. But what about the second principle ? Is the series approach simple and reasonably logical ? Actually it is difficult and frequently dangerous. Let me quote from the preface to Prof. Hardy's *Pure Mathematics*: " I have indeed in an examination asked a dozen candidates, including several future Senior Wranglers, to sum the series $1+x+x^2+\ldots$, and not received a single answer that was not practically worthless ". That was written in 1908 ; if the situation would be materially different to-day, it is largely because of Professor Hardy's own book, and all that I can do is to suggest that some of the ideas therein are applicable as soon as it is thought desirable to talk about limits at all. We can suggest four main difficulties and dangers in the use of the geometric series. First, there is the old, old danger that the pupil may think he is really adding up an infinite number of terms ; this trap is waiting for him whenever you mention an infinite series, and it is wise to defer the meeting until he is wise enough to avoid the danger. Secondly, it is easy to lay too much stress on the series $1+x+x^2\ldots$ with $0 < x < 1$ and not enough on the same series with $-1 < x < 0$. An adequate discussion of the first case tends to leave only enough time to treat the second as a more or less trivial corollary, and this is likely to leave the pupil with a fairly clear notion of the one-sided and monotonic approach to the limit and an idea that there is nothing in the second case which is not in the first. Thirdly, if equal stress has been laid on the two cases, there is still the unfortunate fact that the beautifully regular approach of the partial sums of the geometric series to their limit—the obviousness of the fact that if the nth sum is a good approximation, then all succeeding sums are better approximations to the limit—this is so plain and obvious that it is likely to obscure the really vital point, the imperative need for a similar though less simple behaviour if a limit is to exist at all. What is vital must be separated from what is accidental. If we prescribe a certain degree of approximation, however fine, then all but a finite number of the terms of the sequence must be approximations within this degree ; the geometric series obviously satisfies

this requirement because each partial sum is a better approximation than its predecessor, but this reason is sufficient, not necessary ; to put it crudely, there is a surplus which is accidental to the series, and has nothing to do with the essentials of limit-existence. But danger lurks on the other side ; this is an old point, and I should not have bothered to mention it here, save that it appears in some modern and otherwise valuable books. To say that the variant must get as near to its limit as we please is not enough ; it must ultimately become *and remain* within any given neighbourhood of its limit. Fourthly, the method of approach through the geometric series exhibits the limit as a number to which the variant tends without ever getting there. The approaching is true and important, but not the failure to arrive, since the variant may have taken its limit value as often as it likes. The exponentially-damped sine curve dies away, but it also dies an infinite number of times while it is trying to die. It may be better to travel hopefully than to arrive, but the sequence tending to a limit can do both.

We shall probably hear in this discussion that it is perfectly easy to start limits by the geometric series, and by the exercise of a little ingenuity to guard these loopholes against the entrance of error. That may be true, but a method which has no loopholes to guard would be better ; and I suggest with some diffidence that when so authoritative a document as the Association's *Report on the Teaching of Algebra* outlines the series method and, to my mind, stresses the accidental and obscures the essential features of a limit, then it is time for us as an Association to consider the whole matter and see if it would not be wise to raise the limit from its lowly position at the fag-end of a talk on geometric progressions to one more in keeping with its real importance. The method of teaching limits must be simple, easily grasped, capable of geometrical and numerical illustration, and should be based on ideas which will not have to be displaced should the pupil ever wish to learn even the beginnings of modern analysis. A method satisfying these requirements may not exist, and I have no clear-cut, dogmatic, predestined scheme to offer, only some suggestions which might be worked into any complete scheme.

A good deal of very interesting and instructive work can be carried through on the properties of sequences which tend to zero ; there is no lack of material and there need be no fear that time is being wasted. Not only are such sequences of importance in themselves, but in their behaviour they reveal the essentials of limit-existence more strikingly than the more general case, which nevertheless is little more than a simple corollary to the case of the sequence with limit zero, or the null-sequence as it is now fairly generally termed. Moreover, the crude idea of " tending to zero " can be analysed and then synthesised into the shape of a more or less formal definition without encountering in the process any particularly formidable difficulties of technique. The numerical illustrations, so vital to the beginner's comprehension, are easily

chosen to ensure that the synthesis into a definition, however formal or informal this definition may be, arrives at one which is neither insufficient nor redundant. We must be able to find terms of the sequence as small as we please? But then what of the sequence, $1+1,\ -\frac{1}{2},\ 1+\frac{1}{3},\ -\frac{1}{4},\ 1+\frac{1}{5},\ -\frac{1}{6}\ \ldots$? We must be able to find an infinite number of terms as small as we please? The same sequence, or that annoying one—annoying to beginners because it seems to offend their sense of fair play—1, 0, 1, 0 ... shows that we are still not sufficiently precise. And so the true requirement is educed. Then a valuable exercise can be attacked; a number of simple sequences are written down and considered, and very little ingenuity is required to enable us to form a collection which shall exhibit all the main points, and in addition grow gradually in complexity so that without any sense of strain or artificiality the whole of the relevant technique is acquired. After the very simplest of such sequences have been considered, a varied selection such as $(-)^n/(n+1)^2$, $(5n^2+3n-4)/(n^3+1)$, a^n, na^n, n^2a^n, $\frac{1}{2}\{1-(-1)^n\}$, $(n/n+1)^{n+1}$, $(n/n+1)^{n^2+1}$, is available for the elucidation and amplification of the concept, for the gradual leading up from the crude idea to the formal definition. How far this process is to go will, of course, depend on the judgment of the teacher and the needs of the class, but a great deal of useful information can be acquired concerning not only the main point at issue but also its relations with other parts of elementary mathematics. For instance, by the methods found necessary in dealing with the sequences just mentioned, an attack can be made on the rather more difficult and sophisticated result that the sequence $\left(\dfrac{A_n^2}{B_n^2}-2\right)$ is a null-sequence if $\dfrac{A_0}{B_0}=1$, $\dfrac{A_1}{B_1}=\dfrac{3}{2}$, and $A_{n+1}=2A_n+A_{n-1}$, $B_{n+1}=2B_n+B_{n-1}$; thus we provide a sequence tending to $\sqrt{2}$ which is of course merely the ordinary set of convergents of the continued fraction. But I need not spend more time on this point, as I believe Mr. Dockeray wishes to instance some sequences which he finds of particular value in teaching this subject. Moreover, he will have something to say about geometrical illustrations, which I can therefore safely neglect.

Finally, whatever line of approach be adopted, those sequences which are either steadily increasing or steadily decreasing are worth much more attention than is perhaps usual. The increasing sequence is easier to grasp at first. The initial reaction may be a suggestion that the sequence must go on getting bigger beyond all bounds, but a very slight consideration shows that the phrase "beyond bounds" is unjustifiable. It is an unwarranted addition to the definition, and $(n-1)/n$ demonstrates this. At once this draws attention to the two possibilities. To discuss them, "let the point P move along the line L in a series of jumps, *the motion always being from left to right*. Then either P will pass over the whole line, or its position will gradually approximate to a definite position R on the line L. The theorem is almost intuitive; [the proof given earlier] is merely a

careful analysis of the process of argument implied in but suppressed by our intuition of its truth "—again I quote Professor Hardy. This is again, doubtless, very familiar; but more can be made of its applications, *e.g.* the Newton iteration for a square root. Take a_1 to be any number greater than $\sqrt{c}$. Define $a_2 = \frac{1}{2}(a_1 + c/a_1)$, and so on. Then using nothing more recondite than the fact that the arithmetic mean of two numbers is greater than their geometric mean, we show that the sequence is decreasing, and each term is greater than $\sqrt{c}$. Thus it tends to a limit, which from the recurrence relation must be $\sqrt{c}$. Or we can take the sequence of inverses $b_n = c/a_n$, and show that that is increasing, each term is less than $\sqrt{c}$ and that the limit is the same as that of the a-sequence and so must be $\sqrt{c}$. How easy the actual arithmetic is may be seen in an article by Sir C. V. Boys in the *Gazette* for May, 1932, pp. 111-115, where in a very few lines $\sqrt{10}$ is obtained to 30 places of decimals. But we need not stop here. A slightly more elaborate argument will show that the sequence

$$2^n(x_n - 1), \text{ where } x_n = \sqrt{x_{n-1}}, \ (x_0 = x),$$

is increasing and bounded and therefore tends to a limit, which may either be identified with $\log x$ or even used as a definition of $\log x$. And if we like to go still further, the sequence defined by

$$2^n x_n \quad \text{where} \quad x_n = x_{n-1}/\{1 + \sqrt{(1 + x^2_{n-1})}\} \quad (x_0 > 0)$$

is increasing, bounded, and so tends to a limit, which again we may either identify with arc tan x or use as a definition of this function. I hope that in the course of the discussion Mr. Gibbins will tell us what success he has had in experiments of this kind in his school.

In conclusion, I ought to say that I have purposely quoted from the *first* edition of Professor Hardy's book. It may therefore be that he would now repudiate opinions which he held 25 years ago: but in view of Professor Neville's assertion in his Presidential Address that the schoolboy of to-day is taught very much what his predecessor of 1908 was taught, these opinions, even if they should not carry the sanction of Prof. Hardy's name, may still be relevant and not without value.

Mr. N. R. C. Dockeray (Harrow School): The first introduction to the idea of a limit will probably take place about the age of 16. The boy will then be in his School Certificate year, or, in the case of a really clever boy, in the year previous to that. At such an age his mind will evidently be unable to grasp anything more than the most elementary idea of this difficult subject. The way should therefore be made as easy as possible. He will be at a stage where abstract arithmetical ideas mean little to him without some sort of pictorial representation. I think, therefore, that it is wise to concentrate on examples which can be illustrated geometrically. I shall give a few instances later.

The second point that I wish to emphasise is that examples in which the limit of $f(x)$ as x tends to a is equal to $f(a)$ should be avoided as far as possible. I remember when I was 13 years of age

being warned by a master against supposing that $\infty \times 0 = 1$. He was not dealing with limits at the time, but merely replying to a question put to him by one of his pupils. Unfortunately he did not give any examples to show how $\infty \times 0$ could be anything other than unity, nor did he explain that $\infty \times 0$ is meaningless. The consequence was that I thought he was merely being silly when he issued the warning, and made a mental reservation to believe that $\infty \times 0$ was equal to unity until such time as evidence to the contrary should be forthcoming.

Now I believe that a great many boys have a similar feeling with regard to limits. If you state that $\lim\limits_{x \to 5} x^2 = 25$, and then proceed to explain why, they think that you are merely making difficulties where none exists. But I go further than this, and say that they feel just the same about it if you state that

$$\lim_{x \to 0} \frac{x^2+2x}{x^2+x} = 2,$$

or that

$$\lim_{x \to 1} \frac{x^2+x-2}{x^2-3x+2} = -3.$$

It is true that in these cases, when x is put *equal* to 0 or 1 respectively, the fractions reduce to 0/0, but the operation of dividing numerator and denominator by x or $x-1$ is so simple that the pupil cannot see why it should not be done. It is of no use to try to explain why—he is not ready for that yet.

The fact is that, in the definition of

$$\lim_{x \to a} f(x),$$

the values of $f(x)$ for values of x near to a are considered, but the value when $x = a$ must emphatically not be considered, and it is too much to expect an immature mind to grasp this point.

If we look at the matter graphically we can easily see how we are adding to the boy's difficulties by starting with examples such as those given above. The graph of

$$\frac{2-2x}{x^2-4x+3}$$

is of the form shown in the adjoining diagram, with the exception that the point (1, 1) is missing. This is too much for the average boy of 16 to believe; he thinks you are merely pulling his leg. To him it is as clear as daylight that when $x = 1$, $(2-2x)/(x^2-4x+3)$ is equal to unity, and he begins to suspect that advanced mathematics is not something new and interesting, but merely elementary mathematics with a lot of extra and unnecessary difficulties thrown in.

This example also leads up to the third point I want to make. If we wish to illustrate this limit arithmetically we tabulate values of $(2-2x)/(x^2-4x+3)$ for values of x close to $x = 1$. The values of x

are of course chosen arbitrarily, but so as to approach $x=1$. This imposes a double tax on the pupil's mental grasp. Vaguely he suspects that we are arriving at a limit for $f(x)$ only because we are

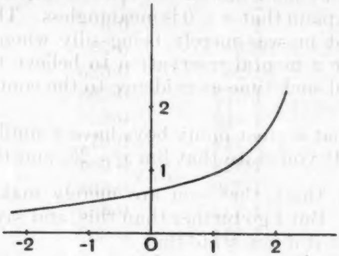


FIG. 1.

making x approach 1 *as a limit.* A glance at such a table will show that there is a certain amount of justification for his suspicion.

x	$f(x)$
0	0·667
0·5	0·8
0·7	0·86957
0·8	0·90909
0·9	0·95238
0·95	0·97560
0·97	0·98522
0·98	0·99010
0·99	0·99502
0·995	0·99752

An example of this sort is of course very useful at a later stage, but it is unsuitable in an introductory course.

Now these difficulties do not arise if we deal at first with functions of a positive integral variable which tends to infinity. But here again we should avoid examples in which the boy is tempted to put $n=\infty$ straight away. Thus a series of exercises of the following type

$$\lim_{n\to\infty}\frac{2n^2+3n+1}{n^2+n+4}=\lim_{n\to\infty}\frac{2+3/n+1/n^2}{1+1/n+4/n^2}=\tfrac{2}{1}=2$$

(since $1/n \to 0$ and $1/n^2 \to 0$ as $n \to \infty$), is almost useless, since it does not in the least help the student to understand the nature of a limit, although it is better than the example quoted previously when treated either graphically or by means of a table. But it is far better to frame examples in which the boy is *compelled* to work out $f(n)$ for successive values of n, and this is most easily done by providing him with a law connecting $f(n+1)$ with $f(n)$. As an illustration, let

$$u_1=0,\quad u_{n+1}=2-\tfrac{1}{2}u_n.$$

We tabulate as follows :

n	u_n	$\frac{1}{2}u_n$	u_{n+1}
1	0	0	2
2	2	1	1
3	1	0·5	1·5
4	1·5	0·75	1·25
5	1·25	0·625	1·375
6	1·375	0·688	1·312
7	1·312	0·656	1·344
8	1·344	0·672	1·328
9	1·328	0·664	1·336
10	1·336	0·668	1·332
11	1·332	0·666	1·334
12	1·334	0·667	1·333

This should also be illustrated graphically. Draw the lines

$$\tfrac{1}{4}x+\tfrac{1}{2}y=1 \quad \text{and} \quad y=x.$$

Then construct the diagram as shown.

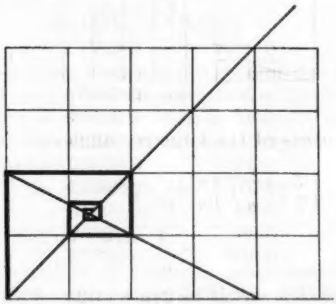


FIG. 2.

Other examples of a similar nature can be dealt with. Thus for the sequence $u_{n+1}=1+\frac{1}{2}u_n$, we draw the lines $y=1+\frac{1}{2}x$, $y=x$. Of course other methods of illustrating the sequence graphically can be used, but the above method should not be neglected, as it is pretty and helps to impress on a boy's mind the nature of a limit. More complicated examples in which curves are used can also be employed, such as

$$u_{n+1}=10+\log u_n, \quad u_1=1.$$

It is clear from this that a boy's first introduction to a limit should *not* be via the differential calculus. Indeed I believe that it is far preferable to treat the integral calculus first; examples can be taken in this field which admit of very easy arithmetical and graphical treatment.

Thus suppose we propound the problem to find the area between the curve $y=\frac{1}{2}x^2$, and the straight lines $y=0$, $x=2$.

Divide the area into n parts by means of the lines $x=2/n$, $x=4/n$, etc. The sum of the areas of the small rectangles is

$$\left\{\sum_{r=1}^{n-1} \tfrac{1}{2}\left(\frac{2r}{n}\right)^2\right\}\frac{2}{n}=\frac{4}{n^3}\sum_{r=1}^{n-1} r^2$$
$$=\frac{4}{n^3}\cdot\frac{(n-1)n(2n-1)}{6}$$
$$=2(n-1)(2n-1)/3n^2.$$

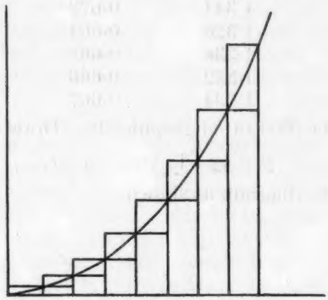


FIG. 3.

The sum of the areas of the large rectangles is

$$\left\{\sum_{1}^{n} \tfrac{1}{2}\left(\frac{2r}{n}\right)^2\right\}\frac{2}{n}=\frac{4}{n^3}\sum_{r=1}^{n} r^2$$
$$=\frac{4}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}$$
$$=2(n+1)(2n+1)/3n^2.$$

The difference between these two is $4/n$, being of course equal to the area of the last big rectangle. This evidently tends to zero as n tends to infinity, and the required area is

$$\lim_{n\to\infty}\frac{2(n+1)(2n+1)}{3n^2}$$
$$=\tfrac{4}{3}.$$

Mr. W. Hope-Jones : I have very little to criticise, in fact (shall I say ?) nothing, in what we have already heard. I agree very cordially with Mr. Broadbent about not waiting till we reach the geometrical progression stage before beginning to learn something about the limit, when that is possible. Unfortunately, under present conditions it is not always so. There is only one statement of Mr. Broadbent's that I contradict : that is, that I am going to make a complete scheme. It never occurred to me to try to make a complete scheme, and I could not if I had tried. And I would add

one word to Mr. Broadbent's statement with regard to finding $\sqrt{10}$ to 30 places on the back of a postage stamp—do not forget to lick the gum off the stamp first!

One of Mr. Dockeray's examples represented something very like a Geometrical Progression; but in general I agree that our present way of taking limits for the first time chiefly as a kind of suburb of Geometrical Progression is not the best way. And yet after we have agreed on that it still remains the fact that most of us are liable to find ourselves called upon to teach Geometrical Progression to boys whom we might describe as quite "unlimited", of whose earlier education we know little or nothing, and who have probably never heard of a limit before; in which case we had better take the opportunity of introducing them to the idea when we can, or for ever hold our peaces!

Before you have time to turn and rend me too savagely for the looseness of my logic, I would remind you of the way in which a skate is put on, and perhaps some other garments as well. Many boots refuse to accept a skate put on tight: you put it on loosely and screw it up afterwards till it holds. I make no attempt whatever at rigorous logic in the early stages. I try to avoid lies that will need to be stamped out afterwards; but you will probably tell me presently that I am not successful even in that.

Then we may take it that we are up against boys learning Geometrical Progression, who have had no introduction to the limit idea; and, as I have said, we have to take the opportunity of bringing in the idea when we can.

Three of the useful illustrations are:

(1) The procession of mice who came and helped themselves in turn from a lump of cheese, each mouse having been so well brought up that it ate only one-third of the cheese that it found on arrival, and left two-thirds of it. An advantage of that illustration is that it is so easy to visualise the remainder, the difference between the sum that you actually have obtained so far and the limit to which that sum is tending. I generally take that first now. Until a year or two ago I used to take Achilles and the tortoise. I think it was Mr. Robson who told me that that ought not to come first, and since then I have taken the mice before it.

(2) The story of Achilles and the tortoise.

(3) Another illustration, which I think is of value because it approaches the limit from both sides, is the shop-assistant who was paid on a profit-sharing basis, getting as part of his wages every year, say, one-fifth of the total profits of the shop for the preceding year, and who broke his employer's window in 1894; how much poorer is he now as a result of that accident?

As I have not time to deal with all of these illustrations, I will say a little about Achilles and the tortoise, though it is the least defensible, and I generally take it after the mice now. I tell you of it chiefly because it was put to me at the age of 11 before I had done

recurring decimals, or even heard of Progressions, and puzzling it out for myself was my first introduction to both. But of course this story is not meant to stand by itself: it requires support from the mice, the shop-assistant, the bouncing marble and a lot of other illustrations.

A Greek philosopher Zeno—not the Zeno who invented Stoics—was fond of catches—unless he meant them seriously, as some people believe now. One was about an arrow which can't move where it is because there is no room in the space it is in for it to move in, and it can't move where it isn't because it can't do anything where it isn't; and so it doesn't move at all. Another catch was about Achilles, who ran ten times as fast as a tortoise and gave him 100 yards start. Achilles ran to the place where the tortoise started, 100 yards; but the tortoise had gone. So Achilles ran another ten yards, to where the tortoise was; but when he got there, the tortoise was—where? Another yard further on. So Achilles ran another yard, but again the tortoise was "not at home". So you go on till you are tired of talking, or of listening, and finally escape by admitting that poor old Achilles never caught the tortoise at all. "Which is absurd."

If Achilles ran x yards before catching the tortoise,

$$x = 100 + 10 + 1 + \cdot 1 + \cdot 01 + \cdot 001 + \dots .$$

"How many terms?" "Oh, I don't know: an awful lot, and then some more." "To infinity?" "Some people call it that; better find out whether it means anything or nothing before arguing too much about that. Call it until tea-time to start with, and leave the lid off in case you want some more."

Then $$10x = 1000 + 100 + 10 + 1 + \cdot 1 + \cdot 01 + \dots .$$

And when you subtract the top line from the bottom you will get $9x = 1000$ to start with. Is that all? One boy in the class will say: "Please, sir, isn't there a sort of hang-over term at the right-hand end?" "Yes, and you're quite right to worry about it, though it doesn't look like being a very big one, does it? If we leave it out we must call the answer approximate at first; but we'll look at it strictly later on and see just how big the error in the approximation is."

So, at any rate approximately, $9x = 1000$, and $x = 111\frac{1}{9}$. Therefore, Achilles ran about $111\frac{1}{9}$ yards before he caught the tortoise. What sort of error is there in this? We can test that by a method that has no suspicion of approximation about it.

While Achilles ran x yards, the tortoise ran $x - 100$; and because Achilles ran ten times as fast as the tortoise, $x = 10(x - 100)$. Therefore $9x = 1000$. Therefore $x = 111\frac{1}{9}$, and that is exact. So the error in the first result is nothing, though it was found by an approximate-looking method for which I did not claim exactness at the time.

This exact quantity is called the limit of the sum of the series. It is an exact answer that seems to have grown out of approximations. If you stop your series before the first term, you get 0 for the

sum, which is a distinctly bad approximation to it: stop it after the 20th and you get a very good one, but still an approximation. The further you go, the smaller is the error; and this raises the obvious question, "If you never stopped your series, would the error disappear entirely?" Without this, or some such verification, I would not trust it to do so. It is a rather dangerous form of words to say, "Add up all the terms that there could be, and you get $111\frac{1}{9}$ for the sum"; but notice that by treating this series rather light-heartedly, as if it just had no end, we have arrived at the limit $111\frac{1}{9}$ as a solution to our problem, and an independent check has proved it to be exactly right, as a solution to the problem at any rate. That is one thing that a limit can do for us.

Quite off the line of what I have said, I would add a word or two as to what Arthur Berry of King's said to me when I went up to King's, having learned at school such things as "$\frac{1}{\infty}=0$", which we used to write quite unblushingly. When I was at school I had very hazy ideas of a limit: perhaps you think them hazy still, but they were more so then. But Arthur Berry said something to me which I have found very useful, especially in teaching the elementary stages of the Calculus: "This business of choosing a value of n which will make the function of n approach as near as you like to the limit can be well thought of in this way. Imagine it as a game or competition between two people. I write down a very small decimal beginning with a great stack of 0's and say, 'Can you choose n so as to make $f(n)$ differ from this particular constant by less than that decimal of mine?' And if you can always choose a value of n that will make this difference less than the small decimal I wrote down, then that constant really is the limit of the function", (though to make it more complete, something would need to be added about the difference *remaining* less than the decimal, as well as *becoming* less).

That is one of the things that Arthur Berry told me, which helped to make my idea of what is meant by a function's approach to a limit rather less fuzzy than it had been before.

Mr. N. M. Gibbins (Central Foundation School), who spoke at this juncture, has expanded the report of his remarks so as to elucidate some points to which he was able to make only very brief reference in his speech, and for this readers are referred to his letter on p. 132 of this number.

Mr. Robson (Marlborough) thought none of the speakers, with the possible exception of Mr. Hope-Jones, had come within two years of the subject! They were discussing "The First Encounter with a Limit", and it seemed to him that the problem which had to be faced was how to avoid the first encounter taking place two years before the stage Mr. Broadbent had spoken about. A great many graphs would be drawn; at any rate, he personally was very fond of setting queer graphs to be drawn, and he supposed that at some time or other an intelligent member of one's class would suddenly

want to know what exactly was meant by the tangent to a curve at a point. How was one to prevent that question coming up a couple of years before the time that one wanted to deal with the subject in the strict sort of way which Mr. Broadbent advocated, and which he (the speaker) approved? He entirely agreed with what had been said as to the desirability of considering other sequences before sequences tending to a limit l, and he also agreed with what had been said with regard to positive integral values. As an example of a trivial elementary limit, he mentioned "$\underset{x\to 0}{\text{limit}}$ of the cost of sending by post a letter of weight $(2+x^2)$ oz."—a rather good example of which probably Mr. Dockeray would approve. Summing up, the speaker said that in his opinion teachers were faced with the difficulty that questions about limits were going to arise in a small way at an earlier stage than Mr. Broadbent contemplated, and he did not think they could be expected to explain exactly what was meant by a limit, at least not at a very elementary stage. It seemed to him that here was a difficulty that had not been met during the course of the discussion, but so far as the suggestions were concerned, he agreed with the cautions put forward by Mr. Broadbent and Mr. Dockeray.

Professor Watson thought if one wanted a function which had zero as a limit, one might use in the third year

$$x \sin(1/x).$$

Mr. M. P. Meshenberg (Tiffin School) offered a suggestion for dealing with the first encounter with a limit on the lines laid down by Mr. Hope-Jones which he agreed represented what one might plausibly expect; his suggestion was to limit the difficulties incident on the use of the words "sum to infinity" and "remainder". The word "infinity", which boys were in the habit of meeting in daily life, had now to be met in a technical sense; that was the cause of confusion. He avoided it by the use of the unorthodox term "sumtoff" of the series. He told the boys that successive sums seemed to approach some number; if by trickery or by any means, dishonest or otherwise, they were able to spot the number which the sums appeared to approach they called that number the "sumtoff". By that device the boy was saved from the use of the word "infinity" and from thinking he had to add up all the terms to get there. He sees also that a "sumtoff" is not a sum.

Professor G. H. Hardy said he had very little to add, because he had never really been concerned with the first approach to the limit. When he used to teach such things he was concerned with the second approach to the limit and, twenty years ago, the problem was extremely simple. Then the one and only thing he had to do was to get his pupils to forget everything that they had previously been taught. Now the situation had changed. Expert opinions with regard to limits had penetrated to the richest and oldest public schools! If he might distinguish between the various

speakers, he would like to congratulate Mr. Dockeray on his exceedingly shrewd analysis of a boy's feelings in regard to the unfairness and unnaturalness of a good many examples with which he was confronted. He had instinctively recalled his own experience when at school. When he allowed for time and for the fact that he was a mathematician and therefore presumably an unusually sophisticated pupil, he was astonished to find that the feelings which Mr. Dockeray attributed to his pupils were almost identical with his own feelings as a boy.

Mr. C. T. Daltry (Roan School, Greenwich), as a member of the Committee which helped to draw up the *Algebra Report*, felt that during the course of the discussion the ordinary boy had been lost sight of. He felt strongly that development in the matter should be psychological and not logical. For instance, the word " limit " was used in so many ways. The ordinary boy might hear the word " limit " used in geometry and trigonometry, in mechanics, physics and other directions—for instance in " speed-limit ". Therefore it was necessary for a boy to receive many illustrations of the limits which occurred, and not till a very much later stage should those various illustrations be analysed in the logical form which had been suggested. Those who had drawn up the Report had tried to produce something of benefit to teachers. He suggested that what had been there submitted was possibly of more benefit to the teacher of the ordinary boy—the boy who would not remain at school after the Certificate stage—than some of the discussion that had taken place that afternoon. While the discussion had been valuable, he would like to see it analysed into much greater detail. How much time was to be given to Mr. Broadbent's treatment, and what was a boy to carry away from it ? At what stage was it to be given ? And so on. The speaker felt conscious of the fact that mathematics was developing, as Professor Neville had pointed out in his Presidential Address, at an enormous rate, and he presumed that the conception of a limit was being analysed more and more, whereas in school mathematics teachers were still dealing with the individual, and it was extremely difficult to bring the fresh knowledge down to that individual's level. With ordinary Higher School Certificate candidates who would never get a scholarship, he had attempted to go through the discussion of the limit by means of sequences and so on, and the ordinary boy asked : " What are you driving at ? " If one gave that boy the sort of illustrations which Mr. Dockeray had mentioned, he said : " I can see that, but where's it leading ? " He had, indeed, found it extremely difficult to convince the ordinary boy of the value of that work.

In conclusion, Mr. Daltry suggested that Mr. Broadbent should elaborate his proposals by means of articles in the *Gazette*, and then teachers would, he felt sure, be only too pleased to see if they could make use of them in practical teaching. But for ordinary practical work, contact with what Mr. Dockeray had suggested was probably the most one could manage for the ordinary pupil.

Mr. G. W. Ward (Archbishop Holgate's School, York) called attention to what he considered an unnecessary complication in evaluating limits. A typical textbook example was the evaluation of

$$\lim_{x \to a} \frac{x^2 - a^2}{x - a}$$

by putting $x = a + h$. A division by h is performed and $2a + h$ obtained. The limit of this as x tends to a, *i.e.* as $h \to 0$, is $2a$. The speaker considered that it was equally valid to divide by $x - a$, because this need not equal 0 as x tends to a. The quotient $x + a$, being identically equal to the fraction considered, has the same limit, namely $2a$.

Mr. T. A. A. Broadbent felt that as there was not time to consider all the topics suggested in the course of the discussion, he need only deal particularly with that one point which seemed to him the most important that had emerged, raised by Mr. Daltry, that all teachers would like to have the simplest possible way of introducing a limit to the schoolboy. The speaker said he disagreed somewhat with the treatment suggested in the *Algebra Report* because it seemed to him too complicated for the ordinary boy: it contained so many things that were irrelevant and superfluous. Could they not, somehow or other, strip off irrelevancies and concentrate on the simple facts of limit existence? That was meant to be the core of the first part of his remarks.

As to the second part, the suggestions were, he confessed, probably beside the point, because he had not the experience of teaching boys which would make those suggestions valuable. He felt, however, that having made the suggestion in the first part of his remarks that a simpler method ought to be found, it would be extremely cowardly to run away from his guns even though he knew they were spiked beforehand.

He would like to say a little more about the treatment in the *Algebra Report*. On p. 71, the series $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} \ldots$ was mentioned, and it was said: "Starting with the 2/3 in our mind and proceeding step by step, we get:

(*a*) alternately beyond and below 2/3;
(*b*) never reach 2/3;
(*c*) get as near to it as we please."

If this was meant to demonstrate the limit-character of 2/3, then it should be noted that the first two conditions are irrelevant and the third, in the form in which it is stated, is not sufficient. To emphasise irrelevancies and miss the vital point seemed to him to be the wrong way to teach boys of any age.

The President thought all would agree with him when he said they were very grateful to Mr. Broadbent and his team for the manner in which they had handled the subject which had just been discussed. Personally, he had never realised how difficult was the problem of teaching limits until he heard the speakers explaining how to do it.

THE USE OF SIGNS IN GEOMETRY.*

By H. V. Mallison.

The idea of attaching signs to certain geometrical magnitudes such as lengths, angles and areas is fundamental in analytical geometry. It has its uses, too, in pure geometry in simplifying certain theorems and constructions. Many theorems, for instance, such as " the area of a triangle is half that of a rectangle with the same base and height ", or " the angle subtended at the centre of a circle is double that at the circumference ", have to be proved differently for two or more figures, the word " add " being in some cases changed to " subtract ". In such theorems the one proof will suffice for any figure provided the proper interpretation is made of the signs of the magnitudes.

It is no doubt debatable when the idea of signs should be introduced in elementary pure geometry. The older textbooks avoided it altogether, even when they came to the converses of Ceva's and Menelaus' theorems ; and the theorem giving one side of a triangle in terms of the other two and the projection of one on the other was divided into two theorems and each written out in full detail. To bring the sign of an angle or an area into elementary work does not appear to have been thought practical until quite recently, in connection with which the use of signs with angles, under the name of "crosses", in Professor H. G. Forder's *Higher Course Geometry*, is very commendable. While the drawing of all the various figures which may arise in a given case is useful in teaching a student to look for and examine every possible alternative, something in the way of paying attention to signs might be done by insisting that segments of straight lines, angles and areas should always be written in a particular way. This principle is used by many teachers in connection with similar figures ; they recommend that two similar triangles ABC, DEF shall be written so that corresponding vertices occur in the same order, so that corresponding sides and angles may be written down without reference to the figure.

To illustrate the point with magnitudes, if A, B, C are collinear and occur in that order, write $AB+BC=AC$, or $CB+BA=CA$; or if C comes between A, B then $AB-CB=AC$. When the use of signs is explained later on, it will be realised that all of these, if correctly written down, are true for any figure. Again, the description of the arm of an angle which came first in a definite sense might be given ; for example, in the triangle ABC, $\angle ABC+\angle BCA+\angle CAB=180°$, in preference to writing down the three angles B, C, A in any way. Also in describing a figure, a definite method of succession of the vertices, say anti-clockwise, might be prescribed ; thus the area of the triangle $ABD+$ area of the triangle ADC is equal to the area of the triangle ABC, if D is on BC. This might be done long before the use of signs is explained, and then when this is known, the

* A paper read at the Annual Meeting of the Mathematical Association, 8th January, 1935.

student will be aware that in cases where there are two or more figures, a proof for any one figure, if carefully written, will suffice for any other, with suitable interpretation of the signs. This is in fact often tacitly assumed to be the case, one figure only being drawn ; it being implied that a similar proof would do for other cases with a few alterations. If, however, the proof is written out with a strict attention to sign, the student will realise better that these " alternatives " are not accidental, but merely other ways of writing the proof.

This method, although it has the appearance of acting rather automatically, must of course be applied with intelligence. Take the following case : A, B are two fixed points on a circle, P a variable point. The fact that the angle APB is constant if P lies on one arc is often extended by taking P at A or B or on the other arc, thus including the theorems that the angle between tangent and chord is equal to the angle in the alternate segment, and that the sum of opposite angles of a cyclic quadrilateral is 180° ; but by no interpretation of sign can we say that the angle APB is the *same* constant for all points on the circle. In the geometrical representation of complex numbers, if A, B, P are represented by α, β, z, no single circle can be represented by $\arg\{(z-\alpha)/(z-\beta)\}=$ constant.

In analytical geometry one is obliged right at the beginning to give an interpretation to sign. Taking plane geometry first, we have two axes prescribed, and positive senses along them. The way in which this is done is purely conventional, but in an elementary treatment it is better to make the positive senses definite. The usual method of taking the axes furnishes a convenient illustration of this. The positive sense along the x-axis, which is horizontal, is from left to right, corresponding to the way a segment AB is written ; the y-axis upwards ; then from x-axis to y-axis gives the positive sense of rotation, that is anti-clockwise, which defines positive angles and areas. Once the senses of the axes are fixed and the signs of the magnitudes with reference to them prescribed, the signs of standard formulae are then determined.

As an instance, the area of the triangle whose vertices are (x_r, y_r), $r=1, 2, 3$, is equal to

$$+\tfrac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix},$$

with the signs described. If the positive sense of the y-axis had been taken downwards, the natural convention of positive area to go with it would be the opposite one ; alternatively, the sign in front of the determinant would have to be changed. The former convention, however, is practically universal, and forms an easy introduction to signs in geometry. The student learns his formulae with a particular sign and knows that they are appropriate to a certain convention. Later on, he will find that it is occasionally more convenient to take other forms of axes, and will be able to tell if there is any change of sign involved in his usual formulae. Instances will readily occur in which the conventional sign is inadvisable ; in

the example given previously, if the sides of a triangle are taken in anti-clockwise order, the internal angles are measured in a clockwise sense to add up to 180° (this consideration is important in the conformal representation of a polygon; see Darboux, *Théorie des Surfaces*, t. I). Again, in polar coordinates it is convenient to take r positive only in the spirals, but positive and negative in the equation of a conic referred to its focus.

In three dimensions, however, the sign conventions are unfortunately much less generally agreed upon. The natural frame of reference is given by the x, y and z axes; and when the x- and y-axes are fixed in the usual way in a horizontal plane to an observer standing on the plane, we may have the z-axis upwards (for right-handed axes) or downwards (left-handed axes). As these axes serve to define other quantities in three dimensions such as screws, products of vectors, torsion, etc., it would seem desirable for simplicity to have a uniform convention. In Professor E. H. Neville's *Prolegomena to Analytical Geometry*, in the early chapters of which the question of attaching signs to geometrical quantities is fully discussed, the two conventions are given, and the statement is made (with which I think most right-handed people will agree) that the right-handed set of axes seems the more natural to take. This is the convention almost universally adopted in vector analysis and mechanics. Originally Hamilton, in his *Quaternions*, adopted left-handed axes as positive, but curiously enough called them a right-handed set; but his pupils, Joly and Tait, both took right-handed axes (in the usual acceptance of the term) as positive.

Unfortunately, for many years it has been customary in analytical geometry of three dimensions to use left-handed coordinate axes. This may have been originally for convenience in drawing figures, as the x-axis then comes in the paper; but the same effect may be obtained with right-handed axes by having the x-axis and the z-axis in the paper and the y-axis perpendicular to them, but in a positive sense away from the reader, as is done in Grieve's *Analytical Geometry*. This has the further convenience that in the case of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, or the hyperboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$, the section by $z=0$ then appears to the reader as a bird's-eye view of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ as usually considered in two dimensions, and its properties are then more readily appreciated by the eye.

Nowadays it is becoming more common to treat differential geometry by vectorial methods, and books which adopt these methods, such as Weatherburn's *Differential Geometry* and Lainé's *Cours d'Analyse*, adopt right-handed axes from the beginning. But even in this case we do not get uniformity, Campbell's *Differential Geometry* using the left-hand system with vector methods.

Before giving further details of the various choices of sign adopted by different authors, I will briefly review the cases where signs may

be in question in an elementary course of analytical solid geometry. The system of signs which I propose is derived from a right-handed set of axes.

Take the origin O in the plane of the paper, the axes OY, OZ in the senses usually attached to the x- and y-axes in plane geometry, and the OX-axis perpendicular to the plane of the paper towards the reader. $O(XYZ)$ then forms a *right-handed* set of axes, such that the spherical representation on a sphere with centre O is a triangle XYZ in which the order of the vertices X, Y, Z is anti-clockwise to an observer outside the sphere. If it is advisable to have the x-axis in the plane of the paper, the axes are rotated through an angle of $90°$ about the Z-axis from X to Y.

If PM is the perpendicular from P to the z-plane, ML perpendicular to the x-axis, OL, LM, MP represent the x, y, z coordinates of P in magnitude and sign, and it is by the projection of these "steps" (W. H. Macaulay, *Solid Geometry*) that most of the formulae for the plane and straight line may be obtained.

If the mutually perpendicular straight lines whose direction-cosines are l_r, m_r, n_r $(r=1, 2, 3)$ can, in this order, be made to coincide with the axes of x, y, z by continuous movement (so that they can be described as forming a right-handed set), then

$$l_1 = +m_2n_3 - m_3n_2, \text{ etc.}$$

According to a term of Mr. W. H. Macaulay's, they are *conformable* with the axes.

The equation of a plane is $lx+my+nz=p$, where l, m, n represent a certain sense along the normal, p the length of the perpendicular from the origin. The quantity p is usually taken as essentially positive, and the sense along the normal that from the origin to the plane. The distance of a point from a plane is then a quantity capable of sign, and hence we can define the positive and negative sides of a plane as the points which when substituted in the left-hand member of the equation of a plane $ax+by+cz+d=0$ make it positive or negative. This is purely a matter of convention, but I should like to suggest a method of defining it which is somewhat contrary to the usual one. When a plane is given, its "perpendicular" form, $lx+my+nz=p$, is completely given in sign as above, and thus the sense l, m, n along the normal, which we may take as positive. We say that the distance of a point (x_1, y_1, z_1) from the plane is *positive* if *the length from the foot of the perpendicular on the plane to the point* is in the positive sense of the normal, and thus the point is on the positive side of the plane. The analytical condition is that the distance of (x_1, y_1, z_1) from the plane is positive if $lx_1+my_1+nz_1-p$ is positive. The more usual way is to measure the distance from the point towards the plane, but it will be noted that the coordinates are measured from the plane to the point, and this latter method of measurement serves as a useful introduction to that of volume.

If p is the perpendicular from O to ABC, the volume of the tetra-

hedron $OABC$ is $\frac{1}{3}p \,.\, \triangle ABC$, and the sign depends on that of p and the triangle ABC. The conventions we have adopted will be uniformly followed if we take p as measured from the base to O, and the area ABC as positive if the sense of description of ABC to an observer at O is positive, that is anti-clockwise, as we are adopting right-handed axes. With this convention, if a plane cuts the axes at A, B, C in the positive octant, p and the area ABC are both negative, and the volume $OABC$ is positive. It follows easily that the volume of the tetrahedron (x_r, y_r, z_r) $(r=1, 2, 3, 4)$ is equal in *magnitude* and *sign* to

$$-\frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}.$$

This is the usual convention, and affords an example of consistence in sign. The factor outside is $(-1)^3/3!$; we have seen that for a triangle the numerical factor in a similar expression is $(-1)^2/2!$; and for the distance between two points A, B on the x-axis, $(x_1, 0)$, $(x_2, 0)$ the expression is $-\begin{vmatrix} x_1 & 1 \\ x_2 & 1 \end{vmatrix}$.

We have thus in the tetrahedron an example of a scheme for positive magnitudes in three different dimensions; A to B, left to right; A to B to C, anti-clockwise; A to B to C to D, right-hand screw.

Right-handed axes conveniently define the motion of a right-handed screw, which is that of a rotation in the positive (anti-clockwise) sense from OX to OY together with a translation in the direction OZ. If an observer stood below the z-plane with a cork-screw along $Z'O$, its point being at O, and turned it from OX towards OY, the point would be driven upwards along OZ. It is the motion along the helix $x=a \cos \theta$, $y=a \sin \theta$, $z=k\theta$ $(a, k > 0)$ as θ increases.

This leads us naturally to the sign of the curvatures of a twisted curve. At a point on a curve there are three principal lines (the *trièdre*): the tangent, principal normal and binormal at the point, to which several equations are referred, and the signs of the equations and various quantities connected with them will naturally be affected by the order and signs of these lines. Here the authorities almost invariably take the tangent, principal normal and binormal conformable with the axes, whichever set has been chosen; but there are two rather important exceptions, namely E. J. Routh in his well-known paper on the subject, and R. H. Fowler in his Cambridge Mathematical Tract, who take the principal normal, tangent and binormal.

The tangent is drawn in the direction of increasing arc, the principal normal towards the concavity of the curve, and the binormal so that the three are conformable with the axes. In connection with this I may mention a change which has been consistently advocated by Professor Neville, namely that the principal

lines, once fixed, should retain the relative positions they take up by continuous variation as the point moves on the curve. This would mean that the radius of curvature is capable of sign, and has many reasons to recommend it ; it is adopted in Macaulay's *Solid Geometry* (p. 184). The next point is in the sign of the torsion, which is given in sign by the second Serret-Frenet equation, $dl_3/ds = \pm l_2/\sigma$, where l_3 applies to the binormal, l_2 to the principal normal. The sign used depends on the principal lines, and the particular convention chosen, and here there is an extraordinary variety of opinion. It began right at the beginning, when Serret chose one sign and Frenet the other. It seems most natural that with the convention we have been discussing, the torsion of a right-handed screw should be reckoned as positive, and this leads to the equation $dl_3/ds = -l_2/\sigma$. This is the convention advocated by Darboux in a note to Vol. IV of his *Théorie des Surfaces*, and in the new edition of Vol. I he has changed all the signs in that volume in conformity with this view. This convention can be well illustrated by a diagram.

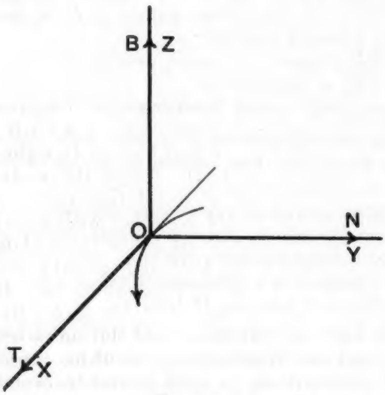


FIG. I.

For positive torsion, the curve twists up into the positive octant. We have then

$$\frac{1}{\rho^2\sigma} = + \begin{vmatrix} x' & y' & z' \\ x'' & y'' & z'' \\ x''' & y''' & z''' \end{vmatrix},$$

where $x' = dx/ds$.

This seems the most simple and natural convention, but, as I have indicated, there is little uniformity in textbooks. By the kindness of Professor Neville, I have been able to consult a large number of textbooks on differential geometry from the library of the Mathematical Association, and I have tabled the results.

The following authors use right-hand axes, and take the torsion

for a right-handed screw as positive, as given in this paper; their second Frenet equation is $dl_3/ds = -l_2/\sigma$.

Wilson: *Advanced Calculus*;
Darboux: *Théorie des Surfaces* (1914);
Weatherburn: *Differential Geometry* (1927);
Blaschke: *Differentialgeometrie* (1924);
Grieve: *Analytical Geometry* (1925).

The following use left-hand axes and take the torsion of a left-handed screw as positive; their second Frenet equation is still $dl_3/ds = -l_2/\sigma$.

Forsyth: *Differential Geometry* (1920);
Smith: *Solid Geometry* (1912);
Niewenglowski: *Géométrie Analytique* (1896);
Frenet: *Calcul Différentiale*;
Campbell: *Differential Geometry* (1926).

The following use left-hand axes, and take the torsion of a right-handed screw as positive; their second Frenet equation is $dl_3/ds = +l_2/\sigma$.

Goursat: *Cours d'Analyse*;
de la Vallée Poussin: *Cours d'Analyse* (1923);
Serret: *Calcul Différentiale*;
Eisenhart: *Differential Geometry.*

The following use right-hand axes, and have the same sign as above, so that they take the torsion of a right-handed screw as negative.

Bell: *Coordinate Geometry of three dimensions* (1923);
Scheffers: *Theorie der Kurven* (1910), following Hermite;
Macaulay: *Solid Geometry* (1930);
Bianchi: *Geometria Diffenziale* (1902);
Lainé: *Précis d'Analyse*, II (1927).

Of writers on vectors, Hamilton used left-hand axes, Tait, Joly and Heaviside (and also Weatherburn) right-hand axes.

The earlier British writers on solid geometry, Salmon, Frost and Price, do not give the Serret-Frenet formula, and do not define the sign of torsion. Fowler and Routh use principal normal, tangent and binormal as conformable with the axes; the former does not define positive σ, the latter uses the positive sign for the second Frenet equation, with left-hand axes. In the note on twisted curves in Vol. IV of his *Théorie des Surfaces*, Darboux uses the fourth convention, following Hermite, but changes the signs in the new edition of Vol. I according to his own preference.

In the American translation of Goursat (1904) there is a note to the effect that right-handed axes are in general use in America, and hence the sign of the torsion would be changed in Frenet's formulae, etc., if the same definition of positive torsion were retained. But, as is seen above, Eisenhart and Wilson, both American authors, use the opposite convention. H. V. M.

CORRESPONDENCE.

THE FIRST ENCOUNTER WITH A LIMIT.

Dear Mr. Editor,—When I read the above title for one of the topics at the Annual Meeting*, I imagined that the speakers would deal with an encounter: a meeting undesigned, unforeseen, perhaps even unwelcome, such as when a boy asks, "Please, Sir, is $\cdot\dot{9}$ equal to 1?"

The earliest encounter mentioned by the openers was the sum to infinity of a geometric progression. Mr. Robson claimed that the earliest encounter was two years earlier than this and instanced the case of the tangent to such a curve as a cubic. I should have put it much earlier. Last term, a class of beginners in geometry were calculating the angles of a regular n-gon for increasing values of n. A boy of 11 asked, "Can it be 180°?"

Occasionally it is interesting to give a class an outline of the history of the evaluation of π. The mention of Archimedes' work brings in the idea of a limit. It is a simple and interesting example for IVth Form boys doing numerical trigonometry to calculate areas and perimeters of regular n-gons for different values of n. Here again π appears as a limit. In the case of the angle of the regular polygon, the class graphed their results and saw the asymptotic relation of $y=180°$. Graphing the results of the trigonometrical work brings in the same idea of the asymptote for π.

Some teachers of authority and experience use the proof by limits of the properties of a tangent to a circle including that of the equality of the angle between tangent and chord with the angle in the alternate segment. Even those who prefer the Euclidean treatment point out to their classes that the Euclidean definition of a tangent to a circle must be abandoned in dealing with tangents to curves in general.

In graphical work, the solution of a set of equations,

$$x^2-2x=0,$$
$$x^2-2x=3,$$
$$x^2-2x+7=0,$$
$$x^2-2x+1=0,$$

by using the graph of x^2-2x, suggests the view that the tangent is the limiting position of a secant.

In compound interest work, questions in which the amount in two years at 4% is compared with the amount in four years at 2% are more interesting than questions with no such connections, and still more interesting are a set of questions in which the amounts of £1 approach the limit e.

I may also add that as an occasional "side-track" with bright IVth Form boys, I obtain

* *See* the Report on pp. 109-123 of this *Gazette*.

$$x=\frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{2+}\,\cdots$$

as the solution of $2x+x^2=1$ and obtain by arithmetical calculation the successive convergents

$$\tfrac{1}{1},\ \tfrac{3}{2},\ \tfrac{7}{5},\ \tfrac{17}{12},\ \tfrac{41}{29},\ \tfrac{99}{70},\ \ldots$$

for $\surd 2$ giving $\surd(2-1)$, $\surd(2-\frac{1}{4})$, $\surd(2-\frac{1}{25})$, $\surd(2-\frac{1}{144})$, ... as convergents with $\surd 2$ as a limit.

All these examples antedate the study of progressions, that is, they are within the range of certificate elementary mathematics. And indeed, for many schools the progressions are not included in the certificate syllabus. In other words, (i) if the first encounter is to be deliberately avoided till the progressions are studied or even later, many pupils will not encounter the idea of a limit at all; (ii) there are opportunities of introducing the idea, and indeed the idea will obtrude itself, despite the teacher's intention to introduce it only at the moment when the pupil is deemed mature enough to study the calculus.

For my part, I welcome those recurrent intrusions which give the teacher an opportunity to clarify and reinforce ideas born in boys' minds and so to give them a background of experience which renders them more receptive when the time comes for rigorous treatment. The small boy's mental vision is often singularly clear and penetrating, and if he does say such things as "the sum of $1+\frac{1}{2}+\frac{1}{4}+\ldots$ is 2", it must not be taken to mean that he thinks the series can be summed, but that he is at a stage of phraseology when he has no better way of saying what he does think.

Those present at the meeting will realise that although the President extended the time for the discussion of the topic, there was not time to bring up these issues.

There is another point to consider. The topic was introduced with the statement that its genesis was due to dissatisfaction felt with the treatment of limits in the *Algebra Report*. Mr. Daltry put in a plea for a statement to be made by competent mathematicians of the precise details in which the *Report* was unsound and of the mode of sound instruction suitable for the certificate stage. I should like to back his plea and ask you to open the pages of the *Gazette* to a discussion of the topic. I do so the more pressingly, as the *Gazette* will reach most of the readers of the *Reports*, whereas only about one-tenth of that number were present at the meeting.

I am, Yours truly,

F. C. Boon.

Sir,—I have to thank you for giving me this opportunity of expanding the remarks * I made at the Annual Meeting of the Association last January.

I take it that the "school certificate" idea of a limit is completely illustrated by the result

* *See* the Report on pp. 109-123 of this *Gazette*.

$$s_n \equiv 1+\frac{1}{2}+\frac{1}{2^2}\dots+\frac{1}{2^n}=2-\frac{1}{2^n},$$

together with the observation that given ϵ, however small, we can always find n so that $2-s_n<\epsilon$. We may look at the matter another way, however. Every member of the steadily increasing sequence

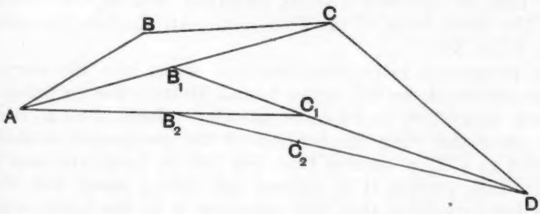
$$1,\ 1+\frac{1}{2},\ 1+\frac{1}{2}+\frac{1}{2^2},\ \dots$$

is less than 2 ; hence a limit exists. Here in point of fact 2 is *the* exact upper bound and so the limit *is* 2. Again, the steadily increasing sequence

$$1+1,\ 1+1+\frac{1}{2!},\ 1+1+\frac{1}{2!}+\frac{1}{3!},\ \dots$$

has *an* upper bound 3 ; hence a limit exists. We can find a smaller bound than 3 if we wish, but we choose the easiest one and that is all that is necessary.

I suggest that the bounded monotonic sequence be taken as the "higher school certificate" idea of a limit. The usual work on convergency can easily be presented in this way ; for example, in strong contrast, the sequence whose general term is $\sum_1^n \frac{1}{r^2}$ has, by appropriate grouping, *an* upper bound 2 ; while the sequence whose general term is $\sum_1^n \frac{1}{r}$ has *no* upper bound. A good illustration from geometry is afforded by the first part of Hobson's proof of the existence of the length of a circular arc, which I have found is not too difficult for the average boy to grasp.



The field engineering method of finding points on the line joining two given positions, neither of which is visible from the other owing to an intervening hill, affords a good illustration of both the above aspects of a limit. Posts are driven into the ground at the given positions A and D, while two observers B and C are also furnished with posts. On C's instruction B moves to B_1, so that C sights B_1A ; then on B's instruction C moves to C_1, so that B_1 sights C_1D. Next B_1 moves to B_2, and so on. Here we have

$$AC+CD > AB_1+B_1D > AC_1+C_1D > AB_2+B_2D > \dots,$$

while every member of this decreasing sequence is independently greater than AD. Hence a limit exists and in this case it is the lower bound, AD, itself. Or we can say that given any small distance ϵ, B and C can move to positions C_n and B_{n+1}, so that

$$AC_n + C_nD - AD < \epsilon$$

or
$$AB_{n+1} + B_{n+1}D - AD < \epsilon.$$

In your review (Vol. XVIII, pp. 285-287) of Adolf Hurwitz' *Mathematische Werke* you call attention to a paper in which he shows how the functions of elementary analysis may be defined by an iterative production of monotonic sequences. I have had the easier parts of this paper translated and have been over the work with my senior boys. In addition to providing a great variety of interesting illustrations of sequences, the "rigidity" of his treatment proved to be an eye-opener to them. I hope to give a précis of the translation in a later number of the *Gazette*.

I am, Sir, Your obedient servant,

N. M. Gibbins.

CAJORI'S EDITION OF NEWTON.

Dear Sir,—With reference to my review of Cajori's edition of Motte's translation of Newton (*Gazette*, XIX, 49), Prof. R. C. Archibald points out that two of my statements need amendment.

(1) It is true that Davis attached no name to the translation of the *De Mundi Systemate* on his title-pages, but in the bibliographical section of the Life of Newton in his edition the ascription is definite (vol. 1, p. liii):

"*A System of the World*, translated from the Latin original; 1727, 8vo.—This, as has been already observed, was at first intended to make the third book of his Principia.—An English translation by Motte, 1729, 8vo."

This paragraph introduces two new dates into the story. The translation which we will agree to call Motte's was certainly issued with the date 1728, and I have no other reference to an edition in 1729. Also the plain implication of the paragraph is that there appeared in 1727 a version that was not in Latin and was not by Motte; this version if it existed has disappeared, but the only reason for supposing that the reference is to the Latin original of 1728 or to a lost edition in Latin is that otherwise the Latin work is omitted from the bibliography.

(2) As everyone knows, "the manuscript in Latin" of the *De Mundi Systemate* which still exists is not Newton's, but a draft in the handwriting of Cotes. It is not disputed that the existence of this manuscript is evidence that the treatise is authentic.

Another mistake is corrected for me by Mr. Zeitlinger. Castiglione says explicitly that the Latin version of the *Method of Fluxions* which he gives is a translation from Colson; the Latin original appeared for the first time, with the title *Geometrica Analytica*, in

the first volume of Horsley's edition, 1779. That is to say, in 1744 this work also " demanded translation into the universal language ".

Yours sincerely,

E. H. Neville.

Reading, 14th February, 1935.

The Editor, *The Mathematical Gazette*.

LAGRANGE'S EQUATION.

To the Editor of the *Mathematical Gazette*.

Dear Sir,—In the *Gazette* for February, 1935, Mr. R. J. A. Barnard, in an article entitled " Lagrange's Equation ", criticizes certain statements which are alleged to appear in my textbook on Differential Equations. May I point out that there is a wide discrepancy between what Mr. Barnard imagined I said and what I really said ? I give two examples in parallel columns :

Barnard.	Piaggio.
P. 31. " Piaggio (p. 147, new edition), begins with the statement that the equations $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$, ...(1) and $Pp+Qq=R$, ...(2) *are equivalent because they represent the same surfaces.*"	P. 147. " We saw that the simultaneous equations $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$, ...(2) represented a family of curves . . . and that $\phi(u, v)=0$ . . . represented a surface through such curves. Through every point of such a surface passes a curve of the family, lying wholly on the surface. . . . Thus equations (1) and (2) are equivalent, for they define the same set of surfaces."
P. 32. " Yet Piaggio calls it a ' Special Integral ', and says that it cannot be deduced from the differential equation or from the given complete integral in the usual way."	P. 150. " It is sometimes stated that all integrals of Lagrange's linear equation are included in the general integral $\phi(u, v)=0$. But this is not so. . . . But $z=0$ satisfies the partial differential equation, though it is obviously impossible to express it as a function of u and v. Such an integral is called special . . . they can be obtained by applying a suitable method of integration to the Lagrangian system of subsidiary equations. . ."

It may be conjectured that the first of these misquotations arose from Mr. Barnard picking out one sentence near the end of my page 147 and ignoring the two paragraphs preceding it. No such

simple explanation can be found in the second case, where the essential word *general* has been altered to *complete*, and a clause has been added which is almost the direct negative of the original. I am sure that Mr. Barnard did not wish to misrepresent me, but it does seem that he must have been a trifle careless in verifying his references. Of course I recognize that the main object of his paper was to plead for a reconsideration of the treatment of special integrals. This is an interesting point which I hope to discuss in a future article.

Yours faithfully,

H. T. H. Piaggio.

University College, Nottingham,
11th February, 1935.

1008. Solution of cross-word, Gleaning 994 (XVIII, p. 344; December, 1934).

[1]2	[2]2	[3]0	[4]4	[5]·	[6]6	2		[7]2	[8]2
[9]3	·	1	4	1	6		[10]1	5	3
[11]4	7	·	9	6		[12]8	4	6	0
[13]0	1	4	1		[14]7	0	6		4
[15]9	8		[16]5	[17]3	1	4	4	[18]1	
	[19]2	[20]2	6	9	8	1		[21]3	[22]6
[23]1	8	1		[24]3	9	3	[25]7	0	1
[26]2	1	7	[27]1	7		[28]4	1	·	0
[29]1	8	5	5	0	[30]8		[31]6	0	2
5		[32]6	7	1	0	8	8	6	4

Across.

7. Methuselah's last birthday-cake had 969 candles (Gen. v, 27).
10. Fishes (John xxi, 11, and heading of chapter).

Down.

10. "Sing a song of sixpence,
A pocket full of rye,
Four-and-twenty blackbirds
Baked in a pie."

If in 1π you bake 24, then in 61π you bake 1464.

20. No. 222 in *Hymns Ancient and Modern* begins "Ten thousand times ten thousand".
27. Acts xxvii, 37; Ps. cxix.

MATHEMATICAL NOTES.

1134. *Two "Lewis Carroll" problems.*

The following solutions to two problems quoted from Lewis Carroll's diary in the *Gazette* of May 1933 may be of interest:

1. To divide any number by 11 or 9 by mere addition and subtraction.

The following methods are based on Horner's method of contracted division and are equally applicable to division by $r \pm 1$ of any number expressed in the scale of notation with radix r:

Divide 8524721 and 542361 by 11.

$$\begin{array}{r} 8524721 \\ 8\bar{3}5\bar{1}8\bar{6} \\ \hline 8\bar{3}5\bar{1}8\bar{6},7 \end{array} \qquad\qquad \begin{array}{r} 542361 \\ 5\bar{1}306 \\ \hline 5\bar{1}306,\bar{5} \end{array}$$

Quotient 774974; remainder 7. Quotient 49305; remainder 6.

METHOD. Proceed as in Horner's contracted process for division by divisor of type $r+1$, placing the first digit under the second and subtracting, etc. Denote negative numbers which occur in the process by the bar notation. This extended use of the bar notation follows the suggestion in § 14 of the Association's *Report on the teaching of arithmetic*. If the remainder given by the last digit on the right is in defect, *e.g.* $\bar{5}$, one must be subtracted from the last digit of the quotient and the true remainder is given by $11+\bar{5}=6$.

Division by 9 may be performed as follows:

Divide 8524721 by 9.

$$\begin{array}{r} 852472\ 1 \\ 84618\ 1 \\ 1\ 1\ 1 \\ \hline 947191\ 2 \end{array}$$

Quotient 947191; remainder 2.

METHOD. Place the first digit of the dividend under the second and add.

(i) If the sum obtained is less than 9, place it under the next digit of the dividend.

(ii) If the sum obtained is greater than or equal to 9, place a 1 under the same column and the excess over 9 under the next digit of the dividend.

Continue the process, using (i) or (ii) as required. Finally, add the three rows so obtained. The right-hand column gives the remainder. If the remainder so obtained is greater than or equal to 9, add 1 to the quotient, the excess over 9 giving the true remainder.

2. The second problem was to find three equal rational-sided right-angled triangles.

Tabulation of a large number of triangles yielded two examples.

(i) 15, 112, 113; 24, 70, 74; 40, 42, 58.

(ii) 120, 182, 218; 56, 390, 394; 105, 208, 233.

A simple but interesting exercise in the theory of numbers also emerged, namely, that the sum of the squares of two odd numbers is never a perfect square. V. I. TODHUNTER.

1135. *The determination of the coordinates of the foci of a curve.*

The use of complex numbers in solving the equations for the foci of a conic (in Note 1113) reminds me of the following method for finding the foci of any curve. It was given me by the late L. J. Rogers, but I have not seen it in precisely the same form in a textbook.

"Write $z=x+iy$, $\bar{z}=x-iy$ and suppose that the equation of the curve becomes $f(z, \bar{z})=0$. Eliminate $\bar{z}$ between $f=0$ and $\partial f/\partial \bar{z}=0$ getting $g(z)=0$. The roots of this equation give the real foci." It should be added that a double point may also be given.

It is easy to see that equation (i) of Note 1113 is obtained if this method is applied to the general equation of a conic—the eliminant is in fact

$$(\partial f/\partial \bar{z})^2=(a-b+2hi)f.$$

In practice in numerical cases it seems a trifle easier to find the centre first. Thus, taking Professor Howland's example

$$8x^2+4xy+5y^2-36x-18y+9=0,$$

the centre, given by $\quad 8x'+2y'-18=0$

and $\quad 2x'+5y'-\ \ 9=0,$

has coordinates $x'=2$, $y'=1$.

The equation referred to parallel axes through the centre is therefore

$$8x^2+4xy+5y^2-18x'-9y'+9=0$$

or

$$8x^2+4xy+5y^2-36=0.$$

On putting $x=\frac{1}{2}(z+\bar{z})$, $y=\frac{1}{2}i(\bar{z}-z)$ this becomes

$$f(z, \bar{z}) \equiv \tfrac{1}{4}(3-4i)z^2+\tfrac{13}{2}z\bar{z}+\tfrac{1}{4}(3+4i)\bar{z}^2-36=0,$$

and so

$$\frac{\partial f}{\partial \bar{z}} \equiv \tfrac{13}{2}z+\tfrac{1}{2}(3+4i)\bar{z}=0.$$

Eliminating $\bar{z}$ we find

$$z^2=-3-4i,$$

so that $z=1-2i$ or $-1+2i$.

Hence the foci are $(x'\pm 1, y'\mp 2)$, *i.e.* $(3, -1)$ and $(1, 3)$.

B. E. LAWRENCE.

1136. *Method for finding the Foci of the General Conic.*

By an elementary theorem of geometrical conics, the tangent from any point on the directrix of a conic subtends a right angle at the corresponding focus. From this it follows that the polar of any point on the directrix passes through the focus and that therefore

the polar of the focus passes through any point on the directrix, *i.e.* is the directrix. Also, the polar of any point on the directrix is perpendicular to its join to the focus. On this property the method chiefly depends.

The locus of a point whose polar with regard to the conic

$$\phi(xy) \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

is perpendicular to its join to the fixed point (p, q) is

$$(x-p)(hx+by+f) - (y-q)(ax+hy+g) = 0, \quad \text{............(1)}$$

i.e. a rectangular hyperbola passing through (p, q) and the centre. If (p, q) is a focus, this must degenerate into the corresponding directrix (the polar of the focus) and a straight line perpendicular to the directrix and passing through (p, q), viz. the axis containing the focus,

$$[x(ap+hq+g) + y(hp+bq+f) + gp + fq + c]$$
$$\times [(x-p)(hp+bq+f) - (y-q)(ax+hq+g)] = 0. \quad \text{.........(2)}$$

Comparing coefficients of x^2 and xy, we have

$$\frac{(ap+hq+g)(hp+bq+f)}{h} = \frac{(ap+hq+g)^2 - (hp+bq+f)^2}{a-b} = k.$$

Considering the values of the left-hand sides of equations (1) and (2) as x and y tend to p and q respectively, the second bracket of (2) becomes identical with (1), showing that

$$k = p(ap+hq+g) + q(hp+bq+f) + gp + fq + c = \phi(pq),$$

giving the well-known equations for the four foci (p, q).

If (p, q) is on either axis, the rectangular hyperbola breaks into perpendicular straight lines. An investigation of this leads to a simple ruler and compass construction for the polar of any point P with regard to a conic whose centre C, focus F and major semi-axis CA are given. If X is the foot of the perpendicular from P to CF and Y, Z two points on CF such that $CX \,.\, CY = CA^2$ and $CY \,.\, CZ = CF^2$, then the straight line through Y perpendicular to PZ is the polar of P. This construction is very simple in the case of the parabola where we have reflexion and where another construction (namely, if X' is the foot of the perpendicular from P to the minor axis and $CX' . CY' = CB^2 = CA^2 - CF^2$, YY' being the required polar) fails.

R. C. Lyness.

1137. *Historical Note.*

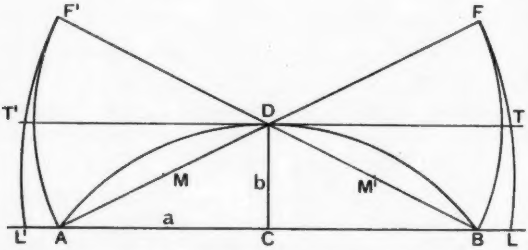
In Mr. Lawrence's novel and interesting article on " Introductory Theorems in Geometrical Conics ", in the *Mathematical Gazette* for October 1934, he refers to the property concerning segments of chords as Newton's theorem. In doing so he followed most other writers on the subject. This property, however, Newton in his *Principia*, Sec. V, expressly refers to Apollonius, Bk. III, Props. 17-23. Apollonius in his general preface tells us that it was new, and that it enabled him to investigate the problem of the *locus ad*

tres et quatuor lineas, of which he gave the solution in Bk. III, Props. 53-56. Strangely enough this solution escaped the notice of Newton, although he was familiar with the *Conics* of Apollonius, and by a different method he worked out a solution for himself which he thought was the first that had ever been arrived at by geometrical methods, and was superior to that previously obtained by analysis by Descartes, who made the problem the basis of algebraic geometry in his work *La Géométrie*. For the conic properties due to Newton, see the article "Newton's contribution to the Geometry of Conics" in *Isaac Newton*, 1642-1727, pp. 96-114.

JOHN J. MILNE.

1138. *Length of a circular arc.*

Huyghens' simple formula for the length of an arc ADB, namely, $(8AD-AB)/3$, where AD is the chord of half the arc, is excellent for calculation but is not very good for a graph, though Mr. T. W. Hall has done his best on p. 275 of the October (1934) *Gazette*.



There is a simpler and slightly more accurate method of depicting the length, either by straightening it out along the tangent at D, or by flattening it out along the chord, either being accomplished by the same construction.

Bisect AD at M and with centre M and radius $3AD/2$ $(=MF)$ describe a circle cutting the tangent at T and cutting AB at L. Then $DT=CL=$half the arc approximately. The other half will of course be given by taking centre M' and radius $M'F'$.

In detail, let $AB=2a$, $CD=b$, then the diameter of the circle is $(a^2+b^2)/b$ and the half arc

$$=\frac{a^2+b^2}{b}\arctan\frac{b}{a}$$

$$=\frac{a^2+b^2}{b}\left(\frac{b}{a}-\frac{1}{3}\frac{b^3}{a^3}+\frac{1}{5}\frac{b^5}{a^5}-\frac{1}{7}\frac{b^7}{a^7}+\dots\right),$$

or, putting $b/a=t$,

$$=a(1+t^2)(1-\tfrac{1}{3}t^2+\tfrac{1}{5}t^4-\tfrac{1}{7}t^6+\dots)$$

$$=a(1+\tfrac{2}{3}t^2-\tfrac{2}{15}t^4+\tfrac{2}{35}t^6-\dots).$$

Now take CL in the figure.

$$\begin{aligned}(\tfrac{1}{2}a + CL)^2 &= \tfrac{9}{4}(a^2+b^2) - \tfrac{1}{4}b^2\\ &= \tfrac{9}{4}a^2 + 2b^2\\ &= \tfrac{9}{4}a^2(1+\tfrac{8}{9}t^2).\end{aligned}$$

Thus $$\begin{aligned}CL &= -\tfrac{1}{2}a + \tfrac{3}{2}a(1 + \tfrac{4}{9}t^2 - \tfrac{8}{81}t^4 + \tfrac{32}{729}t^6 - \dots)\\ &= a(1 + \tfrac{2}{3}t^2 - \tfrac{4}{27}t^4 + \tfrac{16}{243}t^6 - \dots).\end{aligned}$$

Huyghens' formula for the half arc

$$\begin{aligned}&= (4AD - a)/3\\ &= a\{4\sqrt{(1+t^2)} - 1\}/3\\ &= \tfrac{4}{3}a(1 + \tfrac{1}{2}t^2 - \tfrac{1}{8}t^4 + \tfrac{1}{16}t^6 - \dots) - \tfrac{1}{3}a\\ &= a(1 + \tfrac{2}{3}t^2 - \tfrac{1}{6}t^4 + \tfrac{1}{12}t^6 - \dots).\end{aligned}$$

Hence the true value is

$$a(1 + \tfrac{2}{3}t^2 - \tfrac{2}{15}t^4 + \tfrac{2}{35}t^6 - \dots). \qquad \text{(A)}$$

My value is

$$a(1 + \tfrac{2}{3}t^2 - \tfrac{4}{27}t^4 + \tfrac{16}{243}t^6 - \dots). \qquad \text{(B)}$$

Huyghens' value is

$$a(1 + \tfrac{2}{3}t^2 - \tfrac{1}{6}t^4 + \tfrac{1}{12}t^6 - \dots). \qquad \text{(C)}$$

The various coefficients of t^4 are

(A) $\frac{2}{15} = 0{\cdot}133333\dots$;

(B) $\frac{4}{27} = 0{\cdot}148148\dots$; error $= 0{\cdot}0148148 \dots$;

(C) $\frac{1}{6} = 0{\cdot}166666\dots$; error $= 0{\cdot}0333 \dots$;

and the coefficients of t^6 are

(A) $\frac{2}{35} = 0{\cdot}057143$;

(B) $\frac{16}{243} = 0{\cdot}065843$; error $= 0{\cdot}0087$;

(C) $\frac{1}{12} = 0{\cdot}083333$; error $= 0{\cdot}0262$.

So, both for convenience of drawing and for accuracy, my construction is better than one based on Huyghens' formula though the latter is so wonderfully simple for calculation. A. LODGE.

1139. *Small Oscillations of a body with one degree of freedom.*

I feel with Mr. Lowry (*Gazette*, October 1934, Note 1118) that the argument in the textbooks leaves something to be desired, but prefer to modify it as follows.

In a system with one degree of freedom, whose position is determined by a coordinate θ, the kinetic energy is of the form $\frac{1}{2}A(\theta)\,.\,\dot{\theta}^2$, where $A(\theta)$ is positive and not zero for every value of θ through which the system can pass, since the kinetic energy must be positive when the system passes through any position. The energy equation is then of the form

$$\tfrac{1}{2}A(\theta)\,.\,\dot{\theta}^2 + V(\theta) = \text{constant},$$

and if we suppose θ to be chosen so that $\theta=0$ in a position of equilibrium, we shall in general be able to expand $A(\theta)$ and $V(\theta)$ in the forms

$$A(\theta)=A_0+A_1\theta+A_2\theta^2+\dots,$$
$$V(\theta)=V_0+\tfrac{1}{2}a\theta^2+\dots,$$

where A_0 is positive and not zero, and a is positive or negative according as that position is stable or unstable.

Now *while θ is small enough*, the terms $A_1\theta+A_2\theta^2+\dots$ in the expansion of $A(\theta)$ are as small as we please compared with A_0 and we have approximately

$$\tfrac{1}{2}A_0\dot{\theta}^2+\tfrac{1}{2}a\theta^2=\text{constant},$$

and therefore $$A_0\ddot{\theta}+a\theta=0,$$

and hence if a is positive, $\theta=C\sin(nt+\epsilon)$, $n^2=a/A_0$;

or, if a is negative, $\theta=C\sinh(nt+\epsilon)$, $n^2=-a/A_0$.

Both these approximations hold good as long as θ remains small enough, so that, in general, both hold good for a short interval of time during which the system is passing through a position of equilibrium, a stable position in the first case, an unstable one in the second. But if the initial conditions are such that C is small in the first case, the system remains near the equilibrium position and the approximation continues to represent the motion, which is then a small vibration about a position of stable equilibrium.

F. Bowman.

1140. *The Arc of a Hyperbola.*

It has been shown that James Gregory (1638-1675) found the length of an arc of an ellipse by means of the binomial theorem and integration.* By a similar method he found the length of an arc of a hyperbola. The result appears in a letter to John Collins, dated 17th May 1671, and apparently it is the more general form of a previous formula, for it is prefaced by: "Having now room and leisure, I resolve to perfect some things I formerly sent to you. Ye know the series I gave you for the hyperbolic curve serveth only near to the vertex." †

AI and *AS* are the asymptotes of a hyperbola *GEF*. *G* is a point on the curve such that *GD*, drawn parallel to *AI*, is equal to *AD*. *E* and *F* are two other points on the curve, and *EC*, *FB* are parallel to *AI*. *GH* is perpendicular to *AS*.

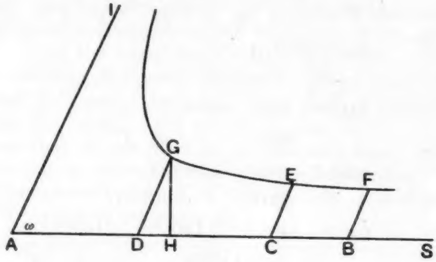
If angle $IAS=\omega$, $AD=DG=r$, $GH=c$, $AC=g$, $AB=f$, and $b=\surd(4r^2-4c^2)$, then $\cos\omega=\pm b/2r$.

* *Math. Gazette*, Dec. 1933, 327-328.

† S. J. Rigaud, *Correspondence of Scientific Men of the Seventeenth Century* (Oxon 1841), ii. 226.

If the equation of the curve referred to its asymptotes as axes is $xy = r^2$,

$$\text{arc } EF = \int_g^f \left\{1 + 2\cos\omega \cdot \frac{dy}{dx} + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}} dx.$$



1. Suppose ω is an acute angle.

$$\begin{aligned}
\text{arc } EF &= \int_g^f \left\{1 - \left(\frac{br}{x^2} - \frac{r^4}{x^4}\right)\right\}^{\frac{1}{2}} dx \\
&= \int_g^f \left\{1 - \frac{br}{2x^2} + \frac{4r^4 - b^2r^2}{8x^4}\right. \\
&\quad + \frac{4br^5 - b^3r^3}{16x^6} + \frac{24b^2r^6 - 16r^8 - 5b^4r^4}{128x^8} \\
&\quad + \frac{40b^3r^7 - 48br^9 - 7b^5r^5}{256x^{10}} \\
&\quad \left. + \frac{64r^{12} - 240b^2r^{10} + 140b^4r^8 - 21b^6r^6}{1024x^{12}} + \ldots \right\} dx \\
&= (f - g) + \frac{br}{2}\left(\frac{1}{f} - \frac{1}{g}\right) + \frac{4r^4 - b^2r^2}{24}\left(\frac{1}{g^3} - \frac{1}{f^3}\right) \\
&\quad + \frac{4br^5 - b^3r^3}{80}\left(\frac{1}{g^5} - \frac{1}{f^5}\right) \\
&\quad + \frac{24b^2r^6 - 16r^8 - 5b^4r^4}{896}\left(\frac{1}{g^7} - \frac{1}{f^7}\right) \\
&\quad + \frac{40b^3r^7 - 48br^9 - 7b^5r^5}{2304}\left(\frac{1}{g^9} - \frac{1}{f^9}\right) \\
&\quad + \frac{64r^{12} - 240b^2r^{10} + 140b^4r^8 - 21b^6r^6}{11264}\left(\frac{1}{g^{11}} - \frac{1}{f^{11}}\right) + \ldots,
\end{aligned}$$

which is Gregory's first formula.

2. Suppose ω is an obtuse angle.

$$\text{arc } EF = \int_g^f \left\{1 + \left(\frac{br}{x^2} + \frac{r^4}{x^4}\right)\right\}^{\frac{1}{2}} dx$$

$$= (f-g) - \frac{br}{2}\left(\frac{1}{f} - \frac{1}{g}\right) + \frac{4r^4 - b^2r^2}{24}\left(\frac{1}{g^3} - \frac{1}{f^3}\right)$$

$$- \frac{4br^5 - b^3r^3}{80}\left(\frac{1}{g^5} - \frac{1}{f^5}\right)$$

$$+ \frac{24b^2r^6 - 16r^8 - 5b^4r^4}{896}\left(\frac{1}{g^7} - \frac{1}{f^7}\right)$$

$$- \frac{40b^3r^7 - 48br^9 - 7b^5r^5}{2304}\left(\frac{1}{g^9} - \frac{1}{f^9}\right)$$

$$+ \frac{64r^{12} - 240b^2r^{10} + 140b^4r^8 - 21b^6r^6}{11264}\left(\frac{1}{g^{11}} - \frac{1}{f^{11}}\right)$$

$$- \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot ,$$

which is Gregory's second formula.

3. When ω is a right angle then $b = 0$, and

$$\text{arc } EF = \int_g^f \left\{1 + \frac{r^4}{x^4}\right\}^{\frac{1}{2}} dx$$

$$= (f-g) - \frac{r^4}{6}\left(\frac{1}{f^3} - \frac{1}{g^3}\right) + \frac{r^8}{56}\left(\frac{1}{f^7} - \frac{1}{g^7}\right),$$

$$- \frac{r^{12}}{176}\left(\frac{1}{f^{11}} - \frac{1}{g^{11}}\right) + \cdot \cdot \cdot \cdot \cdot \cdot \cdot ,$$

which is Gregory's third formula and which he found by substituting $b = 0$ in his first.

ALEX. INGLIS.

1141. *If the bisectors of the base angles of a triangle are equal, the triangle is isosceles.*

The proof of this theorem follows as a corollary to the following: through any point in the bisector of an angle, two, and only two, equal straight lines, terminated by the arms of the angle, may be drawn, and the segments into which the lines are divided at the point are equal, each to each.

Let ACB be any angle and let CO be its bisector. Through P, any point on CO, draw any pair of lines XPY, WPZ in such a manner that $\angle WPC = \angle XPC$.

Then since the triangles XPC, WPC are congruent, $XP = WP$; since the triangles YPC, ZPC are congruent, $PY = PZ$. Hence, adding, $XY = WZ$. Again, since $\angle CXP = \angle CWP$, the points Z, X, W, Y are concyclic. The triangles ZPX, YPW are congruent, hence $ZX = YW$. These are equal chords of the circle $ZXWY$ and

are therefore equidistant from the centre. Thus the centre must lie in the bisector CO, for CO is the locus of points equidistant from CA and CB.

Let the perpendicular bisector of XZ (or WY) cut CO in Q, then Q is the centre of the circle $ZXWY$. Draw the circle and through P draw any chord MN cutting CA, CB in R and S respectively. From Q drop perpendiculars on the same side of CO on to this chord and on to one of the equal chords XY, WZ. Then XY (or WZ) is greater or less than MN according as it is nearer to or further away from Q than MN, and *a fortiori* XY (or WZ) is greater or less than RS. Hence the result.

It follows that if the internal or external bisectors of the base angles of a triangle are equal, the triangle is isosceles. (In the figure "external bisectors" applies to those drawn from B and A to meet CA and CB produced respectively.)

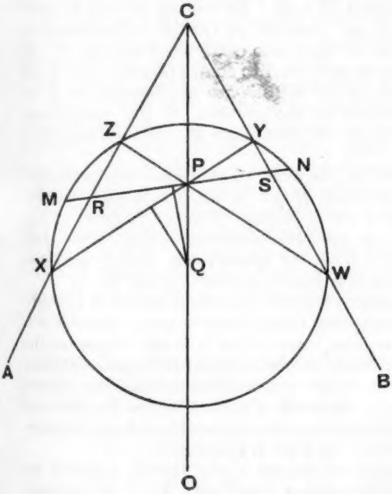


FIG. 1.

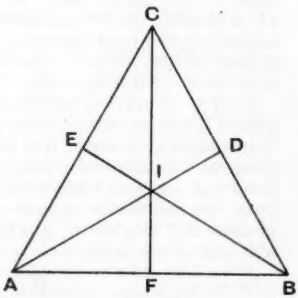


FIG. 2.

For since the bisectors of the angles of a triangle are concurrent, then in the figure the bisectors AD, BE pass through I, a point on the bisector of the angle C. Thus $IA = IB$, and $\angle IBA = \angle IAB$, and so $\angle CBA = \angle CAB$.

Those interested in the theorem are referred to *Nouvelles Annales de Mathématiques* (1842), pp. 138, 311 ; *The London, Edinburgh and Dublin Philosophical Magazine* (1852), p. 366 ; (1853), April, May and June ; (1874), p. 354. The above proof was published in this latter magazine in the December issue of 1913, p. 984.

JAS. W. STEWART.

REVIEWS.

Hypothèse du Continu. By W. Sierpinski. Pp. v, 192. \$3.50. 1934. Monografje Matematyczne, 4. (Warsaw)

The advent of antinomies in the possibility of proving a proposition as well as its negation placed the theory of sets of points in a precarious position. If the same kind of reasoning proves that the proposition a is true as well as that it is false, there must be something wrong with the reasoning or at least with its application to a. Hence the attempts to create a refined logical instrument for mathematical investigation. Russell and Whitehead's *Principia Mathematica* and the efforts of the formalists (Hilbert) are attempts to construct such a logic within the frame of the Aristotelian system. Brouwer thinks that such a logic cannot be applied to arbitrary infinite sets without contradiction and thus tries to limit the sphere of action of the principle of the excluded middle to suit the case.

The scope of the book under review is entirely different. Sierpinski takes the special twofold proposition, called H, that " the continuum can be well-ordered and its cardinal number is $\aleph_1$ " and shows that H is equivalent to various statements about sets, some of them also established without H. In this way we see that H does not seem to lead to contradiction. The book also gives a masterly exposition of all the diverse applications of H in function-theory and also its relation to the problem of the continuum. (Has every non-enumerable sub-set of the continuum the cardinal number c of the continuum ?)

To illustrate the riches contained in this work we give some representative results of the simpler kind. For instance, H is equivalent to saying that the plane is made up of an enumerable set of curves. For the problem of the continuum the most interesting sets are non-enumerable parts of the continuum as little dense as possible. Since we actually can only define such sub-sets of power c, the study of L-sets is most important for our theory.

A set N of power c is an L-set (Lusin set) if the common points of N and any perfect non-dense sub-set of the continuum form a finite or an enumerable set. H is equivalent to saying that there is an L-set. A set N is said to possess the property S if the common points of N and any set of linear (L) measure O form a finite or an enumerable set. The necessary and sufficient condition that a linear set E should contain a non-enumerable S sub-set is that the external measure of E be positive. If a linear S-set is non-enumerable, it is not measurable and no non-enumerable sub-set of an S-set is measurable.

E is a J-set if for every non-enumerable sub-set N of E there is a perfect set in E, having a non-enumerable set of common points with N. The necessary and sufficient condition that a linear set E be a J-set is that E should be the sum of closed sets in number less than $2^{\aleph_0}$.

It follows from H that among Fréchet's linear and non-enumerable dimension types there is no least type.

An $\aleph_\alpha$ is said to be inaccessible if (i) its index α is a limiting transfinite ordinal, (ii) $\aleph_\alpha$ is not the sum of less than $\aleph_\alpha$ transfinite cardinals each of which is less than $\aleph_\alpha$. Every inaccessible aleph is beyond $2^{\aleph_0}$. If this proposition were false, says Sierpinski, c would occupy an almost inconceivable rank.

As applications to function-theory we quote the following results. Every measurable function of a real variable transforms every S-set into a set of the first category, and, conversely, if every measurable function of a real variable transforms the linear set E into a set of the first category, E is an S-set. There is a function of a real variable which transforms every non-enumerable set

into a non-measurable set. There are two linear sets of power c, such that neither of them can be transformed into the other by a Baire function. Every Baire function defined for a linear S-set is of class $\leqslant 1$ in the set. Every Baire function defined for a linear L-set is of class $\leqslant 2$ in the set. There is a function of a real variable discontinuous in every non-enumerable set. It also follows from H that we can "effectively" define (i) a set of functions whose cardinal number is greater than c, (ii) a well-ordered set of functions whose cardinal number is c.

Sierpinski also considers a generalisation of H, namely the statement "there is no cardinal number between m and 2^m, where m is any transfinite cardinal number". It follows then by the principle of selection that

$$\aleph_{\alpha+1}=2^{\aleph_\alpha}.$$

A consequence of the generalised H is that every set of power 2^m is the sum of increasing sets $E_i < E_{i+1}$, such that the power of every E_i is precisely m. This is the generalisation of the statement "H is equivalent to the proposition that the set of real numbers is the sum of increasing enumerable sets".

We have quoted this proposition at the end in order to illustrate the following remark. For the reviewer, the most difficult thing to admit about H is the implication that a process, namely the successive construction of transfinite ordinals, which admittedly produces at every step an enumerable set of ordinals, should be thought of as producing finally a non-enumerable set. Since at every step we cannot put on top more than all the ordinals already produced, the fact that such a step produces an enumerable set looks singularly like a proof showing, as Brouwer would have it, that Cantor's process produces but an enumerable set of ordinals. In other words, it is rather bewildering that the limit of a transfinite sequence of the *same* cardinal numbers ($\aleph_0$) is not the same cardinal number.

This remark is, however, no reflection on our author, who has produced a most valuable monograph, full of his own important contributions, which will remain a standard work on this uncommonly delicate branch of pure mathematics.

P. Dienes.

Fourier Transforms in the Complex Domain. By the late Raymond E. A. C. Paley and Norbert Wiener. Pp. viii, 184. \$3.00. 1934. American Math. Soc. Colloquium Publications, 19. (American Mathematical Society)

If $f(t)$ is sufficiently small at infinity, the function

$$F(x+iy)=(2\pi)^{-\frac{1}{2}}\int_{-\infty}^{\infty} f(t)e^{(x+iy)t}\,dt \quad \text{.............................(1)}$$

is analytic in a strip $-\lambda < x < \mu$, and if, further, $f(t)$ is of integrable square, the integral

$$\int_{-\infty}^{\infty} |F(x+iy)|^2 dy \quad \text{.................................(2)}$$

is uniformly bounded over the strip. Conversely, if (2) is uniformly bounded over a strip, there is a function $f(t)$ satisfying (1) in the "mean square" sense. An important case is that in which $f(t)$ vanishes for $t>0$. In these circumstances $F(x+iy)$ is analytic for $x>0$, the strip becoming a half-plane, and (2) is uniformly bounded there. Conversely, if (2) is uniformly bounded for $x>0$, then there is an $f(t)$ satisfying (1) and vanishing for $t>0$. If now $F(x+iy)$ be transformed by the substitution $x+iy=(1+\zeta)/(1-\zeta)$, which maps the half-plane on a unit circle, we have a connection between a complex

function analytic in a circle, on the one hand, and a real function vanishing over a half-line, on the other. This is one of the devices used by Paley and Wiener in a long series of applications of the Fourier transform methods.

After an introductory chapter they begin by applying the above idea to the theory of quasi-analytic functions, and again to the theory of the " closure " of sets of functions. In the succeeding chapters they apply the Plancherel theory to a number of important integral equations, to the theory of " entire " functions, and to such topics as non-harmonic Fourier series, lacunary series, and generalised harmonic analysis.

The last two chapters deal with " random functions ". A random function $\psi(x, \alpha)$ is defined as the sum in a certain sense of the formal integral of a trigonometrical series

$$\Sigma a_k e^{ikx},$$

where the a_k are a set of random complex numbers. The set of numbers a_k can, for almost all choices, be represented by a parameter α in the interval (0, 1). The functions $\psi(x, \alpha)$ are, for almost all α, continuous and nowhere differentiable. A fundamental property is that, if $\phi(t)$ is arbitrary, apart from certain light restrictions, then

$$\int_0^1 \phi\{\psi(x_2, \alpha) - \psi(x_1, \alpha)\}\, d\alpha$$

is a function of $x_2 - x_1$. Thus " if $y = \psi(x, \alpha)$ is a curve depending on α as a parameter of distribution, the distribution of $y_2 - y_1$ is dependent only on the corresponding $x_2 - x_1$, and not on the ' past ' or ' future ' of x or y. If t is the time, $\psi(t, \alpha)$ represents one coordinate of a particle subject to a random but uniformly distributed sequence of impulses, such as we find in the Brownian motion, according to the famous theory of Einstein and Smoluchowski ". The authors develop the appropriate harmonic analysis, which involves " Stieltjes " integration with respect to random functions (almost never of bounded variation in any interval), and they investigate the zeros of random functions.

The book is written in a vigorous style and, though the subject matter is often difficult, it is quite readable by any one who is familiar with Plancherel's theorem concerning the Fourier transforms of functions of integrable square and the allied technique. The young researcher will be grateful for having so much material still fresh from the experts put within easy reach. Professor Wiener is to be thanked for having completed so quickly and thoroughly the task that he unhappily had to finish alone, and he has evidently taken pains to provide a fitting memorial to Paley. In the preface he says: " I have written elsewhere of the great loss to mathematics by his death; here let me only state the condition in which our joint work was left. Our method of collaboration had been most informal. We had worked together with a blackboard in front of us, and when we had covered it with our joint comments, one or other would copy down whatever was relevant, and reduce it to preliminary written form. Most of our work went through several versions, in writing which both authors took part. Even in that part of the research committed to writing since Mr. Paley's death, it is completely impossible to determine how much is new and how much is a reminiscence of our many conversations ". L. S. B.

Einführung in die höhere Analysis. By E. Lindelöf and E. Ullrich. Pp. ix, 526. Geb. RM. 16. 1934. (Teubner)

This is a German translation of the introductory lectures on Analysis which Professor Lindelöf has given for many years at the University of Helsingfors.

The lectures were prepared with great care in the attempt to make the transition from school-mathematics to university-mathematics as smooth as possible for the student.

The scope and plan of the book will be clear when it is said that the successive chapters deal with: I, The elementary functions; II, Approximations; III, Continued fractions; IV, Limits; V, Derivatives; VI, Length, area, volume; VII, Integrals and applications; VIII, Real numbers and proofs of theorems assumed without proof in earlier chapters; IX, Complex numbers; and there are appendices on determinants.

In Chapters I-VII, a general theorem necessary for the development of the subject, of which the proof depends on the theory of real numbers, is stated without proof and reference is made to Chapter VIII for the proof; for example, in Chapter IV it is necessary to assume the theorem that an increasing sequence has a limit.

Throughout the book the exposition is admirable for its exactness and clarity. The only lapse which I noticed is that the proof given on p. 217 of the rule for differentiating $f\{\phi(x)\}$ may break down if $\phi'(x)=0$. The student who has had the training in mathematical reasoning provided in the first seven chapters should be able to face the difficulties of the Dedekind theory.

Some examples are given; there might well be more. The book is well printed and produced.

J. C. B.

An Elementary Treatise on Pure Mathematics. By N. R. C. Dockeray. Pp. xiv, 566. 16s. 1934. (Bell)

Mr. Dockeray has aimed at producing a textbook of mathematical analysis suitable for scholarship candidates and first-year students at a university. Such students are not yet sufficiently mature to grasp a development of the subject from first principles but the work presented to them should nevertheless be as rigorous as is compatible with their attainments. The result is a comprehensive treatise, going as far as uniform convergence, which will be valuable to teachers of mathematics. The student using the book will require guidance in the order of his reading, as it is not a book which is intended to be read straight through; for instance, Mertens' theorem on multiplication of series appears in the first chapter.

The exposition is, on the whole, accurate, but there are occasional lapses from the high standard of rigour and clarity which a book should set up as a model to the student. I mention a few points—some of little importance—in the order in which they occur in the book.

At the bottom of p. 10, it would be better to say that $|b_n|$ is greater than $|\mu|-\epsilon$ than that the least value of $|b_n|$ is $|\mu|-\epsilon$. The inequality of the arithmetic and geometric means admits, in my opinion, of more interesting proofs than the one given, on p. 13, by induction. The equation in the last line of p. 81 is not necessarily true for principal values of the amplitude, and (two pages further on) the application of de Moivre's theorem to find the addition theorems for cos θ, sin θ, tan θ will probably strike the student as an argument in a circle. On p. 110, the lower bound of an aggregate of numbers is introduced apparently without explanation (or if there is an earlier reference which I have failed to notice, it is not given in the index). In the differentiation of $\phi\{f(x)\}$, the special case of $f'(x)=0$ needs special treatment (p. 123). For the differentiation term-by-term of $\Sigma f_n(x)$, it should be stated that $f_n'(x)$ is assumed to be continuous (p. 464), and for differentiation under the sign of integration that $f_\alpha'(x, \alpha)$ is a continuous function of x and α (p. 476). The convergence of the improper double integral occurring in the expression of the

Beta function in terms of Gamma functions, when x and y are less than 1 (p. 514), is by no means a simple matter.

The book is disfigured by numerous misprints. Some of these are pieces of slovenliness of which no publisher should have been guilty, such as the displacement of the Σ half-way down p. 23, the use of two sizes of type in the statement of the first Pappus theorem (p. 327), and the displacement of the 8 in the numbering of page 383. Other misprints should have been removed in the proof readings, for example: *divergent* is written for *convergent* on p. 21, line 11; n for r (several times) under the sign of summation in the proof of Mertens' theorem (pp. 42, 43); m for $m+n$ on p. 52, line 8; and the signs $\geqslant$ and $\geqslant$ are both used.

One of the best features of the book is the admirable collection of examples, about 1500 in number, of all stages of difficulty. Answers and hints for the solution of the more difficult examples are provided.

Mr. Dockeray has placed his colleagues under an obligation by undertaking what must have been a heavy piece of work. A careful revision for a second edition would increase their debt to him. J. C. B.

Lezioni sulla teoria geometrica delle equazioni e delle funzioni algebriche. IV. Funzioni ellittiche e abeliane. By F. Enriques and O. Chisini. Pp. viii, 274. L. 60. 1934. (Zanichelli, Bologna)

After a considerable delay, Messrs. Enriques and Chisini have completed their great treatise, the first three volumes of which are already well known to the geometrical world. The present volume will be better appreciated as a sequel to the preceding volumes than as an independent treatise on algebraic functions. But as this sequel it makes extremely interesting reading. It is in a sense a comment on the more geometrical theory which precedes it, explaining the function-theory aspects of various questions in the theory of sets of points on a curve. While the general outlines of the usual theory of algebraic functions are given, those questions which are of interest to geometers are considered in more detail than those of purely function-theoretic interest. For this reason the third chapter is the most interesting. In this the theory of multiple theta functions as far as they concern the geometer is described, and then there follows a brief but delightful account of the theory of Abelian varieties, including the Scorza treatment of defective integrals, and the volume ends with a summary of the properties of hyperelliptic surfaces, including Kummer surfaces, with the associated theory of quadratic complexes. It seems a pity, however, that room was not found for an account of the Hurwitz theory of singular correspondences for curves of genus greater than one. In brief, this is a volume which should be in the possession of all who possess the three earlier volumes, and should prove as useful a work of reference for geometers as its predecessors. It is also a pleasure to add that the printing and the paper used are considerably above the standard of certain recent mathematical works published in Italy, and it has been a pleasure to read the bound copy which has been used to write this notice. W. V. D. H.

Mathematische Grundlagenforschung. Intuitionismus. Beweistheorie. By A. Heyting. Pp. iv, 73. RM. 8.75. 1934. Ergebnisse der Mathematik, Band III, Heft 4. (Springer)

The main part of this book contains a summary of the theories of Brouwer and Hilbert on Foundation questions; the logistic position, the decision-problem, and general metamathematical investigations will be treated in another report of this series.

No man can serve two masters, but it is possible, in mathematics at least, to hold to the one without despising the other, even when the masters have been in vigorous opposition; and though Heyting is a disciple of Brouwer, and adopts the intuitionist point of view throughout, he sums up the matters at issue judicially.

If Brouwer's ideas ever become orthodox, the resulting change in mathematics will be far greater than that brought about by the rigorous reconstruction of last century. It is not merely a matter of rejecting an outlying branch like the theory of the higher aleph-numbers, for the radical nature of his doctrine is illustrated by the fact that the following fundamental theorems have to be modified: a bounded monotonic sequence of real numbers has a limit; a continuous function $f(x)$, which has opposite signs at $x=a$ and $x=b$ vanishes for some value of x between a and b; a function continuous in a bounded closed interval attains its maximum. The revision has to go very deep; the relations of "less than" and "greater than" between real numbers are not so simple as in the classical theory. It results that a function everywhere defined in a closed interval is uniformly continuous in the interval. There is no place for the theory of Lebesgue measure.

The development of Hilbert's ideas from the beginning of the century is given in detail, but the Hilbert-Bernays treatise recently reviewed in the *Gazette* was published too late to be considered.

Some attention is also paid to the ideas of the French analysts, to the relation between mathematics and science and to the work of Weyl, Pasch, Kaufmann and Mannoury.

H. G. F.

Kristallprojektion im Vergleich mit entsprechenden Erdkarten und mit einer Anwendung auf die Laue-Aufnahmen. By W. HEINTZE. Pp. iv, 32. Kart. RM. 1.20. 1934. Mathematisch-Physikalische Bibliothek, Reihe I, Band 82. (Teubner)

It is clear that the whole of practical crystallography cannot be condensed into 32 pages, and the author wisely does not make the attempt. He confines himself to stereographic, gnomonic, orthographic and reflexion projections, in which a point of a sphere with longitude θ and north polar distance ρ is mapped on a plane by a point with vectorial angle θ and with radius vector $\tan \frac{1}{2}\rho$, $\tan \rho$, $\sin \rho$, $\tan 2\rho$ respectively. And, indeed, this is about as far as he carries mathematical formulae. For the most part theorems such as "stereographic projection is conformal and maps circles by circles" are merely stated, and the reader is left to supply a proof for himself, or obtain it from some larger textbook. But the nature of the four projections is described clearly by means of 27 very ingenious and attractive diagrams. In these are shown the map of the lines of latitude and longitude on the earth when the plane of projection is perpendicular or parallel to the polar axis, and the map of the poles of a beryl crystal on a plane perpendicular to the hexagonal axis of symmetry. To those who read German with ease the book can be recommended as a very pleasant little introduction to a fascinating topic.

HAROLD HILTON.

Das Spiel der 30 bunten Würfel. MacMahons Problem. By F. WINTER. Pp. 128. Kart. RM. 3.60. 1934. (Teubner)

There are thirty different ways in which a cubical die may be painted with six given colours, so that each face has a different colour. With a set of thirty coloured dice a large number of elegant arrangements are possible, many of which are described in detail in this book. A typical problem considered is the following: one of the dice being given, it is required to select eight of the

remaining ones and from them to construct a die of twice the linear dimensions of the original one, which shows the same colours on its external faces, such that any two internal faces which are in contact have the same colour. The eight dice are uniquely determined, but can be arranged in two different ways. The other problems considered are of much the same type.

In the later chapters some games possible with the dice are described. The variant of dominoes in Chapter XV seems attractive and not too complicated, and we are impelled to express our admiration of the illustrations in this chapter, particularly the swastika on p. 122.

In spite of the description of the book, on its cover, as *ein mathematischer Zeitvertreib für jedermann*, we can't help feeling that the average English Everyman will find the book hard going. There is much entertainment in the book, however, for the mathematician. The projective geometer, for instance, will not fail to notice the prominent part played by the fifteen divisions of six letters into three pairs, and will be inevitably reminded of a familiar configuration in four dimensions !

The volume is excellently printed, and the diagrams are very clear. J. A. T.

Wie findet und zeichnet man Gradnetze von Land- und Sternkarten ? By G. Scheffers. Pp. 98. Kart. RM. 2.40. 1934. Mathematisch-Physikalische Bibliothek, Reihe I, Bd. 85/86. (Teubner)

The chief thing to be said against this booklet is that when RM. 2.40 is translated into English the result appears to be 3s. 9d. At the German price it must be good value for the money, for it provides in 98 pages an account of map projections which is surprisingly complete.

It is stated that the book is intended for those who understand school mathematics, but this phrase does not seem to imply a knowledge of the calculus, for calculus notation is used only in connection with Mercator's projection, and even then the integration is not given. The emphasis throughout is on graphical methods for constructing the various projections, approximate constructions being given where necessary.

The historical aspect is adequately treated, not in the rather self-conscious form of appended notes, reminding the reader, as it were, of an interesting sideline, but in a natural way as an integral part of the subject.

The section on star maps is brief, being practically confined to the construction of a planisphere. E. H. L.

A Compact Arithmetic. By W. G. Borchardt. Pp. vii, 295, xxxviii. 3s. 1934. (Rivington's)

This book is best described, in the words of the preface, as an adaptation of the author's *First and Second Course in Arithmetic*, for those who require a shorter book. Its existence testifies to the fact that the style and content, by now well known, have found favour with a large number of teachers. Although shorter, it is nevertheless complete and extremely thorough. There are a very large number of examples and plenty of revision papers. A set of four-figure tables is included. E. H. L.

Relativity Physics. By W. H. McCrea. Pp. vii, 87. 2s. 6d. 1935. Methuen's Monographs on Physical Subjects. (Methuen)

The object of this little book is to collect together such portions of the special theory of relativity as are of importance in physics, excluding the general theory on the ground that its observable consequences have so far been entirely astronomical. Only elementary mathematics is used, but sometimes

steps are omitted which a weak mathematician may find difficult to supply. Vector notation is introduced merely for brevity; the theory of vectors is avoided, and tensors are wholly excluded. The book is stated to be intended in the first place for physicists, but in all references to observations merely the results are quoted without experimental details. The description of the Michelson-Morley experiment is extremely brief; perhaps the author expects students to have read of it elsewhere. It is stated that the effect was negative, and that all the observational evidence from other sources substantiates this. This statement occurs in most expositions of relativity, but it is not entirely accurate. Dayton C. Miller (*Reviews of Modern Physics*, Vol. 5, July, 1933) claims to have established, as a result of a long series of observations on Mount Wilson, that there is an ether-drift effect with an apparent value of ten kilometres per second. It is quite possible that this claim may have to be rejected, but it should not be ignored. An interesting article by Dr. C. V. Drysdale on this subject will be found in *Nature* (24th November and 1st December, 1934).

The earlier chapters of Professor McCrea's book deal with the Lorentz transformation, kinematics, and mechanics, all of which have been so thoroughly explored for many years that it is difficult to say anything new about them. The later chapters, dealing with optics, electromagnetic theory, atomic physics, thermodynamics, statistical mechanics, and hydromechanics, are of more value. There are paragraphs on such recent topics as Milne's cosmology, Dirac's wave-equation, aurorae, cosmic rays, nuclear transformations, stellar radiation, Synge's theory of the annihilation of particles, and Heisenberg's uncertainty principle. Thus the book furnishes, at a very low price, a treatment of various topics usually found only in much larger and more expensive works.

H. T. H. P.

Chemical Kinetics and Chain Reactions. By N. SEMENOFF. Pp. xii, 480. 35s. 1935. International series of monographs on physics. (Oxford University Press)

Chemical kinetics entered on a new phase of its existence with the appearance of Hinshelwood's book in 1926. The subject has since grown greatly in beauty and attractiveness; and physicists as well as mathematicians have not concealed their partiality for it. The present book—the first to appear on the subject—has been written by a physicist, who by confessing to some difficulty in matters of pure chemistry, successfully disarms any potential critics from among the chemists. The confession, however, is as unnecessary as it is graceful, for there are few chemists who will read this book without benefit and delight.

Professor Semenoff has himself made some of the major contributions in this field; and the treatment, while taking full advantage of independent developments made at Oxford, Berlin and elsewhere, is based largely on work carried out in the author's laboratory at Leningrad.

The book is divided into four parts, roughly equal, dealing with general theory, reactions of the halogens, oxidations, and miscellaneous instances of decomposition and polymerisation. In all, nearly fifty chemical reactions are dealt with descriptively and analytically. While conceding that the amount of detail to be included is a matter of taste, it is regrettable, for numerous and fairly obvious reasons, that the story has not been told more briefly, particularly when this could have been done without loss of effect.

We are told at the outset of the author's belief that chemical kinetics requires complete reconstruction from the standpoint of chain mechanism. There will be no necessity to remind the type of reader for whom this book

is written that there are other beliefs. The author's opinion may be not altogether dissociated from the fact that there has been much omission of reference to collateral work. This would have merited inclusion on account of its relevance, which is incontestable, if not of its importance, which is admittedly a matter of personal judgment. In view of this omission, it is not altogether surprising that in his general survey (p. 462) the author has drawn one thoroughly misleading conclusion. This in no way detracts from the otherwise uniformly reliable standard of the work. The subject-matter, which is a difficult one, has been presented lucidly and in an attractive manner.

Regarding the work " in the gross and scope of its encompass ", Professor Semenoff is to be warmly congratulated for compiling an accurate, stimulating and almost exhaustive account of chemical chain reactions. The present volume can take its place with dignity among the other members of a distinguished series.

The format is up to the unsurpassed standard which one has long learnt to expect from the Oxford University Press. E. A. Moelwyn-Hughes.

Sur les groupes des transformations analytiques. By H. Cartan. Pp. 53. 14 fr. 1935. Actualités Scientifiques et industrielles, 198; exposés mathématiques, ix. (Hermann, Paris)

This work is a study of general groups of analytic transformations in several complex variables, the aim of the author being to determine conditions for such a group to be continuous in the sense of Lie, that is, whether it is possible to determine the parameters of the group in such a way that the law of composition of the elements is analytic in the parameters. J. A. T.

La méthode du repère mobile, la théorie des groupes continus et les espaces généralisés. By E. Cartan. Pp. 66. 12 fr. 1935. Actualités Scientifiques et industrielles, 194; exposés de géométrie, v. (Hermann, Paris)

This tract gives an account of a generalisation of the idea of moving axes, familiar in elementary differential geometry, to a form which can be adapted to study the intrinsic geometry of spaces of various kinds. The subject is intimately connected with the Lie theory of continuous groups, since each such group may be regarded as a set of transformations in a space, the space being the group itself. The principle of the method is to associate a configuration with each point of the space, with the property that there is just one displacement in the space which changes the figure associated with a point A into that associated with any other point B. The equations determining the local variations of these figures can be determined, and are intimately related to the structure-equations of the group. These equations, in fact, contain in essence the whole of the differential geometry of the space. Several examples are given in illustration. J. A. T.

Évariste Galois et la Théorie des Équations algébriques. By G. Verriest. Pp. 1-58. 1934. (Louvain; Gauthier-Villars)

This booklet contains first an account of the life and early death of Galois, then a résumé of the theory of algebraic equations as it was known prior to his investigations. The section on Galois' theory extends to 28 pages (31-58), and only deals with the simplest equations to which his methods are applicable. In fact the only equations mentioned are those of the biquadratic and binomial types

$$x^4+px^2+q=0, \quad x^n-a=0.$$

The account of the Galois groups is of the simplest character.

Professor Verriest's book can be recommended to a reader wishing to have a brief introduction to Galois' theory as it was known to Galois. It gives no reference to any later work, and on the ground covered, there is much more in Jordan's account of the subject in *Math. Annalen,* I. The book is disappointing in that it makes no mention of such simple resolvents as the quadratic resolvent of a cubic or Euler's reducing cubic of a general quartic, and apparently the author did not wish to trespass on work more recent than that of Galois himself. W. E. H. B.

Elementary Practical Mathematics. I. (First Year). By E. W. GOLDING and H. G. GREEN. Pp. viii, 160. 5s. 1934. (Pitman)

The title " Practical Mathematics " is one which is used in at least two somewhat different senses. On the one hand there is that group of teachers who present the subject through experiment and measurement with class-rooms or laboratories equipped with necessary apparatus for the special work of cutting out, making and/or measuring areas and solids, and developing model instruments for use in connection with the early work in trigonometry. To these the word " Practical " means experimental work in a laboratory from which results are obtained for use in the development of mathematical knowledge. On the other hand there is that group of teachers who by the term " Practical Mathematics " imply mathematics which is used by the practical worker, namely, the engineer, the builder, and the like. It is upon this latter interpretation that this book is written, for in their preface the authors write that they " have aimed at producing a tool for direct application to elementary practical problems ".

It is assumed, and probably rightly so, that the student is familiar with the more practical side from his daily employ. His need of mathematics is that he may understand the theory of his work and be able to make use of it in dealing with the formulae which he has constantly to handle.

The book under review is the first of three which the authors are producing, one for each of the three years of the National Certificate Course. This, Book I, may be said to start from the beginning of mathematics, for Chap. I opens with units and assumes only a knowledge of the four elementary rules of arithmetic, and takes us through fractions, decimals—including a section on recurring decimals—squares and square roots. One is pleased to see stress laid upon the wisdom of the rough check, a short section being specially devoted to this.

Chap. II is a very clear exposition of the interpretation of formulae, but there seems to be an error on p. 30 in what would have been otherwise a good section on " Calculations involving errors of measurement ". It seems that the authors misread 11·7 for 11·3 in an original manuscript and carried on the argument based upon 11·7 and " 1·88333 as 1/6 of 11·7 ", for at one point is the statement " although it is true to say that 11·7 divided by 6 gives 1·88333 ". . . .

Algebra, which is the subject of Chap. III, is introduced by means of problems and equations therefrom. It is a very comprehensive section. The method recommended for the factorisation of an expression as $6x^2-7x+2$ is to call the expression E and then

$$\begin{aligned} 6\,.\,E &= 6\,.\,6x^2-7\,.\,6x+12 \\ &= 2\,.\,3\,.\,(3x-2)(2x-1)\,; \\ \text{thus}\quad E &= (3x-2)(2x-1) \end{aligned}$$

does, as is stated in the next paragraph, " contain a step in which errors are frequently made ", namely, to multiply both factors by the 2 . 3 outside.

A full chapter is devoted to logarithms with a more than passing reference to the slide-rule. Similarly the sections on Trigonometry (including geometry) and Mensuration are presented in that lucid manner which should make any student capable of handling, transforming and evaluating formulae, using logarithm and other tables, and working out his results at work and elsewhere with a reasonable feeling that he is going to be accurate.

The final section deals with graphs and their interpretation. A recommendation of the *Algebra Report* (p. 40) reads " The convention that the values of the independent variable are always plotted from left to right across the page must be emphasised from the first ". In the example of the men walking in opposite directions (p. 126) one feels that convention has given way to a possible convenience.

At the end of the book is reproduced tables of logarithms, antilogarithms, natural sines, cosines and tangents from Godfrey and Siddons tables.

On the whole the book is very good. The print is very clear. If Books II and III are as good for their purpose as Book I is for the beginners, the three volumes should form a very useful series. E. J. A.

Advanced Practical Mathematics. By W. L. Cowley. Pp. xii, 272. 15s. 1934. (Pitman)

As far as we know, Mr. Cowley is not a regular teacher of mathematics, and we wondered whether his book gained in attractiveness for this reason. At all events, a glance through it is sufficient to invite a closer inspection, which at once makes the reader feel that he is face to face with the kind of calculation that must force itself daily upon the designer of engines and structures and other workers in the field of engineering mathematics, the kind of calculation that often calls for a high degree of skill when, for example, it is necessary to extract numerical results, accurate enough for their purpose, from an intractable mathematical formula or differential equation. From this point of view the book has a more practical, and in places a more elementary, aspect than some other books on numerical mathematics, and will recommend itself to those engaged on engineering problems.

The opening chapter is on dimensional theory because, as the author explains in the preface, " the dimensions of formulae should be studied before an extensive numerical reduction is carried through " ; out of this theory arises the principle of dynamical similarity, and applications to deducing the behaviour of ships and aircraft from models are given. After this novel but appropriate beginning follow chapters on the numerical evaluation of formulae; the determination of laws to curves, with stress on the method of least squares; the solution of algebraic equations with real or complex roots ; calculations by finite differences ; numerical integration and the numerical integration of differential equations. A feature is a chapter on matrices, again with the needs of the computer strictly in view, particularly in problems of stability and vibration, and by way of illustration the theory is applied to calculating the periods of the normal modes of vibration of three simple pendulums in cascade. A list of integrals and series is displayed in an appendix.

The author has apparently confined himself to the point of view of the computer, and to the methods that he has no doubt found most serviceable himself. He has tried to make the book self-contained, and so has avoided references, but we feel that there must be many pages that will excite the curiosity of the reader, and that a sprinkling of references would have been an improvement. We should have welcomed also a few examples on which the reader could exercise himself and gain confidence in applying the methods advocated in the text.

By way of criticism it may be said that, while we do not look for rigour in a book of this kind, such loose statements as those on the convergency of infinite series on p. 166 are to be deprecated. The printing is good, but there are a number of misprints, particularly in the appendix, which seems to have been compiled rather hastily. Thus, several of the integrals listed involve a parameter, but there is no indication of the range of the parameter for which the result quoted is valid; again, the range of validity of the power series in θ for arc sin θ is stated to be $-1/\surd 2 < \theta < 1/\surd 2$! F. B.

The Elements of Analytical Geometry. III. Conic sections. By J. T. Brown. Pp. 169-325, vii. 2s. 6d. 1934. (Macmillan)

After a preliminary introduction of the conic by the focus-directrix property, there are separate chapters on the parabola, ellipse, and hyperbola. These are written on the usual lines, but with some improvements in presentation as compared with the older textbooks; they contain about 370 examples.

There is a chapter on $l/r = 1 - e \cos \theta$ which is not particularly original, and a short chapter on tracing conics from the general equation. There is the usual casual treatment of "infinity" in geometry, and it is implied that a circle is a special case of a conic because it is a limiting case of an ellipse.

There are 200 revision exercises at the end. The ground covered is that of Higher Certificate Pass papers and the book would be quite suitable for that work, especially as Higher Certificate examiners seem to regard Conic Sections and Analytical Geometry as the same thing. A. R.

Hydrostatics and Mechanics. By A. E. E. McKenzie. Pp. x, 272. 3s. 6d. 1934. (Cambridge)

Elementary Mechanics and Hydrostatics. By F. Barraclough. Pp. vii, 282. 3s. 6d. 1934. (University Tutorial Press)

Both these books cover approximately the same ground, giving an elementary course in Hydrostatics and Mechanics up to School Certificate standard. They include work on Density, Pressure (in liquids and atmospheric) and the Principle of Archimedes in Hydrostatics, and in Mechanics they cover the usual ground up to and including a little about Momentum and Energy. Each book has examples at the ends of the chapters, including some School Certificate questions. The treatment is elementary throughout, and such difficulties as "Limits" are carefully avoided, thus introducing points in the treatment of velocity and acceleration which those who are particular might not consider satisfactory. Both books contain numerous pictures and illustrations which greatly add to the pleasure of using them, besides explanatory diagrams and interesting historical information.

The first of the two books is divided into two sections, the first containing Hydrostatics and the second Mechanics: it is the first of three volumes on Elementary Physics. One of its features is the number of interesting descriptions of things of everyday life, such as water-wheels, balloons and bridges, with plenty of photographs and diagrams: the teacher with this book will have no excuse for not showing the applications of the work to real life, or for not making it interesting. At the end of each chapter is a summary of the work of the chapter: gravitational units are used at first and absolute units are introduced later.

The second book is not divided into sections, but the order of treatment is Statics, Hydrostatics, Dynamics. Not so much space is given to descriptions of things (though there are plenty of applications to everyday life), and the space saved is given up to more detailed explanations, while there are numer-

ous worked examples, so that the book could be used by the pupil working on his own. Absolute units are used at first to avoid the transition from gravitational units. J. W. H.

Compound Interest Tables. By W. A. Forster. 5s. 1934. (Cambridge)

These tables are a reprint from those which appeared in the *Journal of the Institute of Actuaries*, vol. lxv, pp. 365-401, and give the values of v^n, $(1+i)^n$ and $v^{n+\frac{1}{2}}$ for rates of interest varying from 2 per cent. up to 6 per cent. by $\frac{1}{4}$-intervals and for values of n from 0 to 105.

All those engaged in actuarial calculations will be provided with some table giving the v^n and $(1+i)^n$ values, such as Oakes, Archer, or McKie, but the table of $v^{x+\frac{1}{2}}$ is not usually available and for this function alone it will be worth while securing a copy. In the old days the work was usually done by logarithms, and then Thoman's Table could be used, but now machinery has made the function itself more useful than its logarithm. The type is small but very clear and beautifully printed and on one side of the page only. While interest tables are generally constructed for commercial purposes, it should not be overlooked that they form useful geometrical progressions, and therefore deserve general consideration. W. S.

The Poetry of Mathematics, and other essays. By D. E. Smith. Pp. v, 91. Paper, 50 cents; cloth, 75 cents. 1934. *Scripta Mathematica* Library, 1 (*Scripta Mathematica*, New York)

In this stimulating series of essays Professor Smith makes a plea for the unity of human interests. The relationship of the fine arts to mathematics, or of the social sciences to mathematics, are illustrated convincingly in the various chapters of this book: the Poetry of mathematics, the Call of mathematics, Religio Mathematici, Thomas Jefferson and mathematics, Gaspard Monge, politician. Professor Smith has some pertinent remarks to make about the teaching of mathematics in the schools, which he considers to be far from perfection. The universal value of mathematics is brought forth by the author in his allegory of the " Seven Lamps of the Capella Pittagora ", which his imagination visited in the small hamlet of San Mathesis. There the meaning of the seven lamps was explained to him, being in turn the lamp of utility, of beauty, of imagination, of poetry, of mystery, of infinity, and of religion. As the author rightly observes, if mathematics may not make any man more religious, it makes him see the grandeur of religion as nothing else can, provided he is religiously inclined. T. G.

Lessons in Elementary Analysis. By G. S. Mahajani. Second edition. Pp. xii, 264. Rs. 5. 1934. (Aryabhushan Press, Poona)

Professor Mahajani tells us that his book is intended as a text for the analysis which forms the base of the calculus course for the B.A. degree in Indian universities; geometrical applications of the calculus and differential equations are omitted. He has made considerable use of Cambridge lecture notes and of the treatises by Bromwich and de la Vallée Poussin. The result is a clear, precise and readable account of the subject.

The author deals of course with irrational numbers, limits, continuity, derivatives and integrals and their properties, infinite integrals, uniform convergence and related topics. There is also an introductory account of Fourier series. It is surprising that the theory of the exponential and logarithmic functions is not given a chapter, and that readers are expected to pick this up from examples, worked and unworked, occurring here and there in the text.

The chapters on mean value theorems for derivatives and for integrals are very good, though the proof of the second mean value theorem for integrals might be still further simplified by making the first step the proof that

$$\int_a^b f(x)\varphi(x)\,dx = \int_a^\xi f(x)\,dx,$$

if $f(x)$ is integrable and $\varphi(x)$ monotonic, decreasing and such that $\varphi(a)=1$, $\varphi(b) \geqslant 0$. The usual forms are then easily deduced. The gain is perhaps not great, but it is worth while in dealing with a theorem which causes some difficulty to beginners simply because the wealth of detail to be taken into account does often obscure the main trend of the argument. The simpler proof of the result which assumes the existence and continuity of $\varphi'(x)$ is also given, a useful addition since this form suffices for some elementary applications. We are glad to see the author's own investigation of a general form of remainder in Taylor's theorem, which he first gave in the *Messenger* some years ago; this form includes all the usual forms as special cases.

There are naturally a good number of misprints, but not more than might be expected. If a third edition is called for, some points might be considered: the nature of the Dedekind definition of a real number is carefully explained, but the average student would profit by a more detailed exhibition of its manipulation in the ordinary processes of algebra: the verbal definition of a limit is dangerously vague; given ε, $|a_n - l|$ must not only " become ultimately less", it must ultimately become and remain less than ε. The treatment in symbols is perfectly sound.

The examples are well chosen, but not always well expressed; for instance, to write simply (p. 108)

" Show that $$\lim_{x\to\infty} \frac{\varphi(x)}{x} = \lim_{x\to\infty} \{\varphi(x+1) - \varphi(x)\}"$$

is to encourage those habits of careless thinking and expression which it should be the first duty and aim of a course such as this to eradicate.

T. A. A. B.

Enzyklopädie der Elementarmathematik. I. Arithmetik, Algebra und Analysis. By H. Weber and J. Wellstein. 5th edition prepared by P. Epstein. Pp. xvi, 582. Geb. RM. 20. 1934. (Teubner)

The reader unfamiliar with this book should perhaps be told that the word Encyclopaedia is used in a somewhat modified sense. The book is not a classified collection of all results which may be said to be elementary, nor is it an enumeration of the methods of elementary mathematics. It is an attempt, based on a really prodigious amount of labour carried out with a truly Teutonic thoroughness, to give the reader an opportunity of relating any really important branch of elementary mathematics—that is, roughly speaking, mathematics up to the beginnings of the calculus—to the logical foundations as well as to the appropriate non-elementary extensions of that branch. Thus, for example, the complex number has its elementary geometrical significance fully expounded, while the formal algebraic development, starting from the definition of a complex number as an ordered pair of real numbers, is made to lead naturally towards the investigation of quaternion and vector algebras where the commutative law breaks down.

The first section of the volume deals with arithmetic. The integers are defined by the Frege-Russell method, and then the number domain is extended to embrace negative numbers, fractions, irrationals and complex

numbers. Powers and logarithms naturally come in when roots are mentioned. Then permutations and combinations lead to the binomial theorem, and the section closes with two chapters on the theory of numbers.

Algebra here means the study of algebraic equations, and is developed naturally up to the proof of the algebraic insolubility of the general quintic equation, while the division of the circle and numerical methods for solving equations are taken on the way—though Horner does not get mentioned.

The analysis section is essentially an introduction to limits, in the form of a treatise on the exponential function. Logarithmic computation by means of the comparison of an arithmetic progression with a geometric progression forces consideration of a base

$$e_n=\left(1+\frac{1}{n}\right)^n$$

which tends to a limit as n tends to infinity, and then it is proved that

$$\lim_{n\to\infty}\left(1+\frac{z}{n}\right)^n \quad\text{and}\quad \lim_{n\to\infty}\left(1+z+\ldots+\frac{z^n}{n!}\right)$$

exist and are identical. A great many related topics are considered, and finally the natural apex of all this domain is reached in a proof that e and π are transcendental.

But the outstanding feature of the book is the tremendous amount of historical information conveyed by the matter in small type and in footnotes; the latter are numerous, varied, and full of historical and bibliographical details. The authors have allowed themselves very few asides, but one may be quoted. It refers to the flood of "proofs" of Fermat's Last Theorem evoked by the announcement of the Wolfskehl prize, and goes on: "Sie haben den Fermatschen Satz nicht bewiesen, wohl aber einen beschämenden Tiefstand der allgemeinen mathematischen Bildung, und sie bieten kein mathematisches, sondern nur ein kulturgeschichtliches und psychologisches Interesse."

The 4th edition (1920) and the present 5th edition have both been revised by Dr. P. Epstein. His work has been by no means superficial, for the table of contents is marked by asterisks which show that of the 139 paragraphs now contained in the book, more than half are either new or completely rewritten versions of the original. Not only have methods once fashionable been superseded, but results which were in 1903 neither elementary nor widely appreciated, are in 1935 perhaps still not elementary but have so permeated the whole of mathematics that their impact and influence on elementary work is considerable.

The printing is of the high order of excellence which we invariably expect in a Teubner book. T. A. A. B.

Tables de logarithmes des nombres et des fonctions trigonométriques à quatre décimales. By S. de Glasenapp. Pp. 126. 6 fr. 1934. (Gauthier-Villars)

The logarithm tables give the logarithms of the numbers from 1 to 999 to four places, and of the circular functions for intervals of 1′; the natural functions are tabulated at intervals of 10′. There are no tables of differences. Intended for the practical man, obliged to calculate at a distance from his desk or his machines, the booklet is made for the pocket—in fact, for the waistcoat pocket, since the page measures $3\frac{1}{2}$ inches by 5 inches. T. A. A. B.

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LONDON BRANCH.

At the Annual Meeting on 26th January it was reported that membership and the financial balance had remained satisfactory. The new President was Dr. P. B. Ballard, and the new Chairman, Mr. G. L. Parsons. The meeting then proceeded to a vigorous discussion of the *Algebra Report*.

On 23rd February a provocative paper on "The Teaching of Mathematical Analysis in Schools" was read by Mr. N. R. C. Dockeray of Harrow. It is hoped that this paper, and possibly some of the points raised in discussion of it, will appear later in the *Gazette*. C. T. Daltry, Hon. Sec.

QUEENSLAND BRANCH.

Twelfth Annual Report, presented on 20th April, 1934.

The Annual Meeting was held at the University on 31st March, 1933: the Annual Report and Financial Statement were presented and adopted and the officers for the ensuing year elected. The Presidential Address by Professor Simonds was on the subject "Approximate Integration".

During the year three ordinary meetings were held. At the first, held at the Boys' Grammar School on 2nd June, 1933, Mr. T. Rimmer, M.Sc., read a paper entitled "Modern Meteorology—a branch of Dynamics". At the second, held on 4th August, also at the Grammar School, Mr. E.W. Jones, B.A., read a paper entitled "Glimpses of Reality"; and at the third, held at the University on 27th October, Mr. J. P. McCarthy, M.A., read a paper on "Ancient Methods for the Duplication of the Cube".

The number of members is 26, of whom 11 are members of the Mathematical Association. Copies of the *Gazette* come to hand regularly and are circulated among Associate Members. The Statement of Receipts and Expenses shows a credit balance of £4 9s. 6d., an increase of 6s. 8d. on that of last year. The high rate of exchange on subscriptions to London is still a financial handicap. Attendance at meetings has been maintained, and I feel grateful to those members who respond generously when asked to submit papers for meetings.

Officers: *President*: Professor E. F. Simonds; *Vice-Presidents*: Mr. S. Stephenson, Mr. I. Waddle; *Hon. Secretary and Treasurer*: Mr. J. P. McCarthy; *Committee*: Miss E. H. Raybould, Miss E. M. Cribb, Mr. E. W. Jones, Mr. R. A. Kerr, Mr. J. C. Deeney.

J. P. McCarthy, Hon. Secretary.

YORKSHIRE BRANCH.

The spring meeting of the branch was held on 9th February, 1935, in the Physics Department of the University of Leeds. There were 35 members and friends present, including several of the physics staff of the University.

The following motion, proposed by Lt.-Col. E. N. Mozley, was passed:

"That this Branch considers that the time allowed for the mathematical papers of the Common Examination into Public Schools is insufficient and requests the Association to confer with the Headmasters' Associations on the question".

The Associations mentioned include both the Headmasters' Association and the Headmasters' Conference. Lt.-Col. Mozley will provide further evidence when required.

Professor Hartree gave a very interesting lecture on "The mechanical integration of differential equations". This was illustrated with lantern slides and was much appreciated. Professor Brodetsky proposed a vote of thanks to Professor Hartree, and Mr. Gabriel a vote of thanks to Professor Whiddington for allowing the meeting to be held in the Physics Department.

J. D. Edington, Hon. Sec.

JOURNALS RECEIVED.

When no number is attached, no part has been received since a previous acknowledgment.

Abhandlungen aus dem Math. Sem. der Hamburgischen Universität.
American Journal of Mathematics. 56 : 4 ; 57 : 1.
American Mathematical Monthly. 41 : 7, 8, 9, 10 ; 42 : 1, 2, 3.
Anales de la Sociedad Científica Argentina. 117 : 3, 4, 5, 6 ; 118 : 1, 2, 3.
Annales de la Société Polonaise de Mathématique.
Annali della R. Scuola di Pisa. Ser. 2. 3 : 3-4 ; 4 : 1, 2.
Annals of Mathematics. Ser. 2. 35 : 4 ; 36 : 1.
Berichte über die Verhandlungen der Akad. der Wiss. zu Leipzig : Math.-Phys. Klasse. 86 : 2, 3, 4.
Boletin Matematico.
Boletin Matematico Elemental.
Boletín del Seminario Matemático Argentino. 3 : 14.
Bollettino della Unione Matematica Italiana. 13 : 4, 5 ; 14 : 1.
Bulletin of the American Mathematical Society. 40 : 9 i, ii, 10, 11, 12 ; 41 : 1.
Bulletin of the Calcutta Mathematical Society. 25 : 4 ; 26 : 1.
Communications . . . de Kharkoff. Ser. 4. 8 ; 9 ; 10.
Contribución al Estudio de las Ciencias Físicas y Matemáticas.
L'Enseignement Mathématique. 33 : 1-2.
Ergebnisse eines Mathematischen Kolloquiums (Wien). 6.
Esercitazioni Matematiche (Catania). Ser. 2. 8 : 1-2, 3.
Gazeta Matematica. 40 : 1, 2, 3, 4, 5, 6, 7.
Jahresbericht der Deutschen Mathematiker-Vereinigung. 44 : 1-4, 5-8, 9-12.
Japanese Journal of Mathematics. 11 : 1, 2, 3.
Journal of the Faculty of Sciences, Hokkaido. Ser 1. 2 : 3.
Journal of the Indian Mathematical Society. N.S. 1 : 2, 3.
Journal of the London Mathematical Society. 9 : 4 ; 10 : 1.
Journal of the Mathematical Association of Japan. 16 : 5, 6 ; 17 : 1.
Mathematical Notes.
Mathematics Student. 2 : 2, 3.
Mathematics Teacher. 27 : 6, 7, 8 ; 28 : 1, 2, 3.
Meddelanden från Lunds Univ. Mat. Seminarium. 1 ; 2.
Monatshefte für Mathematik und Physik.
Nieuw Archief voor Wiskunde.
Periodico di Matematiche.
Proceedings of the Edinburgh Mathematical Society. Ser. 2. 4 : 2.
Proceedings of the Physico-Mathematical Society of Japan. Ser. 3. 16 : 9, 10, 11, 12 ; 17 : 1, 2.
Publicaciones . . . Físico-Matemáticas . . . de la Plata.
Publications de la Faculté des Sciences de Masaryk. 191, 193, 194.
Revista de Ciencias (Peru).
Revista Matemática Hispano-Americana (Madrid). Ser. 2. 8 ; 9 ; 10 : 1-2.
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